The Complexity of Free Combinations of Temporal CSPs

Johannes Greiner

(joint work with Manuel Bodirsky)

Institute of Algebra, Technical University Dresden

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- Goal: complexity classification of infinite domain CSPs by means of decomposition of signatures

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- Approach: bottom up, i.e., combine weak structures (and possibly their algorithms) into richer ones
- Choose a combination with "minimal interaction", i.e., only equality and disequality of variables are shared
- Focus: problems in P (tractable) vs. NP-hard problems

Definitions

Definition (CSP, classical)

$\Gamma \dots$ structure with finite relational signature τ

CSP(Γ):

- Input: A first-order sentence ϕ over τ, using only $\wedge$ and $\exists$ (*primitive positive* sentence)
- Output: Is ϕ satisfiable in Γ?

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Definition (CSP, generalization)

$T \ldots$ set of first-order sentences over signature τ

CSP(T):

- Input: A primitive positive sentence ϕ over τ.
- Output: Is there a model for $T \cup \{\phi\}$?

Examples

Example (classical CSP)

Structures with first-order definition over $(\mathbb{Q}; <)$ are called *temporal languages*. The complexity of their CSPs has been classified by Bodirsky and Kara '10.

Examples:

- $\mathrm{CSP}(\mathbb{Q}; (x = y < z) \vee (z = x < y) \vee (y = z < x))$ is in P
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Example (CSP of a theory)

Let T_1 and T_2 be the theory of $(\mathbb{Q}; \{0\}, \neq)$, each theory having its own predicate symbol for $\{0\}$.

There is no structure Γ such that $\mathrm{CSP}(\Gamma) = \mathrm{CSP}(T_1 \cup T_2)$.

First-Order Expansion

Definition (reduct, first-order expansion)

Let Γ be any structure with signature τ. For $\sigma \subseteq \tau$ we define the *σ-reduct* of Γ, written as Γ^σ, as the structure obtained from Γ by forgetting functions and relations which are not in σ.

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If Δ is a reduct of Γ, then Γ is an *expansion* of Δ.
If furthermore all functions and relations in Γ have first-order definitions in Δ, we call Γ a *first-order expansion* of Δ.

Result

Theorem (Bodirsky, G.)

Let Γ_1 *and* Γ_2 *be first-order expansions of* $(\mathbb{Q}; <, \neq)$ *with disjoint, finite relational signatures and* T_1, T_2 *the first-order theories of* Γ_1, Γ_2 *respectively. Then* $\mathrm{CSP}(T_1 \cup T_2)$ *is in P if both* Γ_1 *and* Γ_2 *have tractable CSPs and binary injective polymorphisms, and NP-hard otherwise.*

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Example (Classification in action)

- $T_i = \text{Theory}(\mathbb{Q}; <_i, \neq, R_i,)$ for $i = 1, 2, 3, 4$, where

  $R_1(\bar{x}) := \{\bar{x} \in \mathbb{Q}^4 \mid (x_1 \neq x_2) \vee (x_3 >_1 x_4)\}$,

  $R_2(\bar{x}) := \{\bar{x} \in \mathbb{Q}^3 \mid (x_1 >_2 x_2) \vee (x_1 >_2 x_3) \vee (x_1 = x_2 = x_3)\}$,

  $R_{3,4}(\bar{x}) := \{\bar{x} \in \mathbb{Q}^3 \mid x_1 = \min_{<_{3,4}}(x_2, x_3)\}$
- CSP$(T_1 \cup T_2)$ is in P, CSP$(T_3 \cup T_4)$ is NP-hard.

Tractable Case

Theorem (Nelson, Oppen '79)

Let T_1, T_2 be stably infinite, tractable, ***convex*** *theories with disjoint, finite relational signatures including $\neq$.*
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Definition (convex)

A theory T is *convex* if for any pp-formula ϕ the following holds:

$$T \cup \{\phi\} \vdash \bigvee_{i=1}^{n} x_i = y_i \quad \Rightarrow \quad \exists i : T \cup \{\phi\} \vdash x_i = y_i$$

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Lemma

Let Γ be an ω-categorical structure with finite relational signature where $\neq$ is pp-definable. Then Theory(Γ) *is convex iff Γ has a binary injective polymorphism.*

Free Combination

Definition ($*$-operator)

For disjoint relational signatures τ_1, τ_2 and classes of finite τ_1, τ_2 structures $\mathcal{K}_1$, $\mathcal{K}_2$ we define

$$\mathcal{K}_1 * \mathcal{K}_2 := \{S \mid S^{\tau_1} \in \mathcal{K}_1 \text{ and } S^{\tau_2} \in \mathcal{K}_2\}.$$

Free Combination

Lemma

Let T_1, T_2 be ω-categorial theories with disjoint, finite relational signatures τ_1, τ_2 and without algebraicity.
Then there exists an (up to isomorphism unique) model Γ of $T_1 \cup T_2$ with countably infinite domain D such that

$$\mathrm{Sym}(D) = \overline{\mathrm{Aut}(\Gamma^{\tau_1}) \circ \mathrm{Aut}(\Gamma^{\tau_2})} = \overline{\mathrm{Aut}(\Gamma^{\tau_2}) \circ \mathrm{Aut}(\Gamma^{\tau_1})}. \qquad (\dagger)$$

The structure Γ is ω-categorical, without algebraicity and $\mathrm{Age}(\Gamma) = \mathrm{Age}(\Gamma^{\tau_1}) * \mathrm{Age}(\Gamma^{\tau_2})$.

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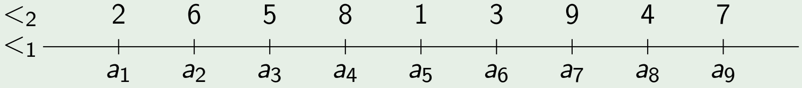
We call any model of $T_1 \cup T_2$ with $(\dagger)$ the *free combination* of the models of T_1 and T_2 respectively.

Free Combination of Temporal Structures

Example (Free combination)

Two copies of $(\mathbb{Q}; <)$ have a free combination $(\mathbb{Q}; <_1, <_2)$ with two independent orders (has been studied by Cameron, Linman and Pinsker and others).



Hard Case

Reminder: Γ_1, Γ_2 FO expansions of $(\mathbb{Q}; <, \neq)$, Γ free combination of Γ_1, Γ_2.

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   2. convexity is equivalent to: For any pp-sentence ϕ and variables x_1, x_2, x_3, x_4: $\Gamma \models \phi \wedge x_1 \neq x_2$ and $\Gamma \models \phi \wedge x_3 \neq x_4$ then $\Gamma \models \phi \wedge x_1 \neq x_2 \wedge x_3 \neq x_4$.

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    4. take essentially binary poly. f of Γ with witnesses u, v.

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   5. there are witnesses u', v' of essentiality of f in the same orbit as $(s_1(x_1), s_1(x_2), s_1(x_3))$ and $(s_2(x_2), s_2(x_3), s_2(x_1))$ respectively

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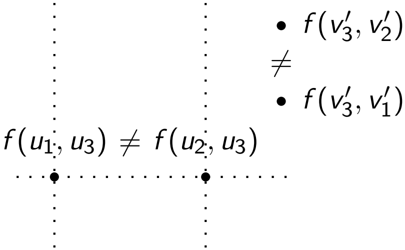
   6. hence there are $\alpha_1, \alpha_2 \in \mathrm{Aut}(\Gamma_1)$ s.t. $f(\alpha_1, \alpha_2)(s_1, s_2)$ is a solution for $\phi \wedge x_1 \neq x_2 \wedge x_3 \neq x_4$

2.5 in Detail

Remember: f preserves $<_1$ and $<_2$.



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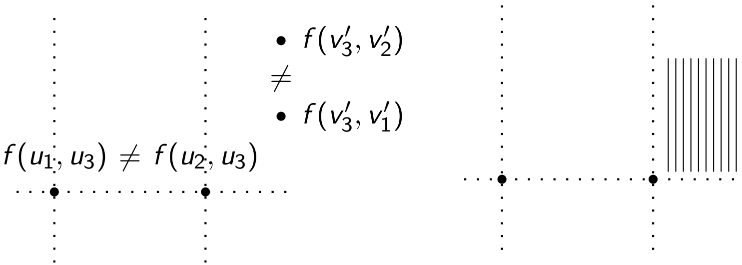

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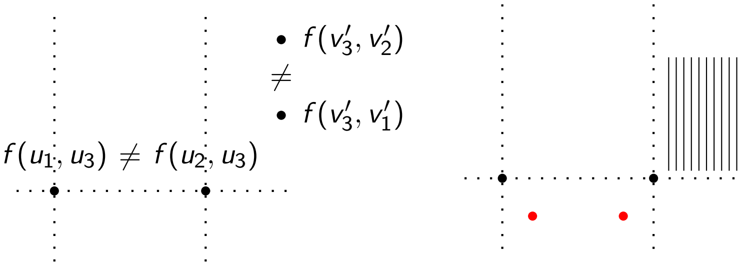

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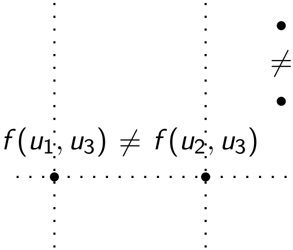

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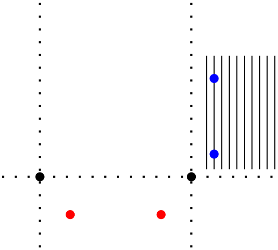

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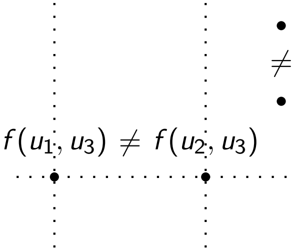

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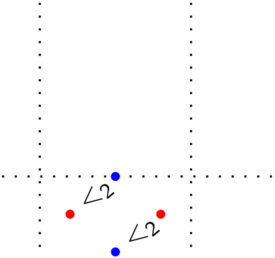

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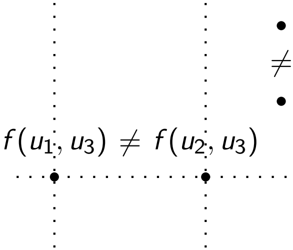

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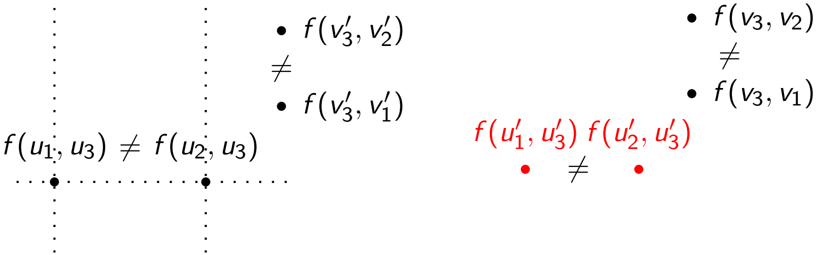

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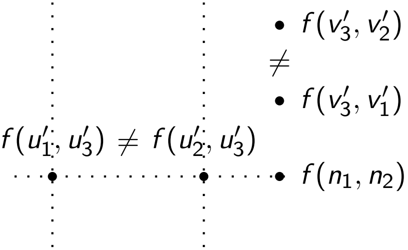

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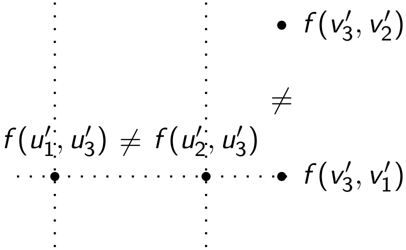

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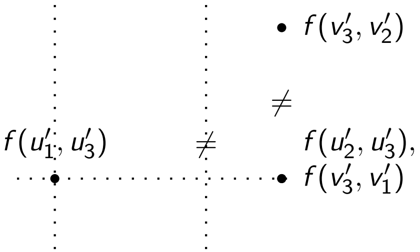

Questions?

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