On a combinatorial problem in directed graphs

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AAA94 & NSAC 2017
17 June 2017

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Notations

$G = (V, E)$ - a (finite) graph, with vertex set V and edge set E
$\mathbb{N}$ - the set of nonnegative integers
$[a]_\sim$ - equivalence class of an element a with respect to an equivalence relation $\sim$
$N(x)$ - set of (direct) neighbours of the vertex x in the graph G
$N^{-1}(x)$ - set of inverse neighbours of the vertex x in the directed graph G
$\mathcal{D}(G)$ - set of distribution functions on G

Distribution functions on a graph

Definition

For a finite graph $G = (V, E)$, a function $f : V \longrightarrow \mathbb{N}$ will be called a distribution function on G. For any $S \subseteq V$, the weight of S with respect to the distribution function f, denoted $w_f(S)$, is the nonnegative integer

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Redistribution of a distribution

Definition

Let $G = (V, E)$ be a graph, and $f, g \in \mathcal{D}(G)$. We say that f can be redistributed into g if there is a function $r : E \longrightarrow \mathbb{N}$, called redistribution function, such that

$$f(x) = \sum_{y \in N(x)} r(x, y), \forall x \in V$$

and

$$g(y) = \sum_{x \in N^{-1}(y)} r(x, y), \forall y \in V.$$

We shall write $g = r \cdot f$ in this case.

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Ordering distributions and associated distributions

Definition

Let $G = (V, E)$ be a graph, and $f, g \in \mathcal{D}(G)$. We say that f precedes g, and denote this $f \preceq g$, if $f = g$ or there is a finite sequence of redistribution functions $r_1, r_2, \ldots, r_k$ and of distribution functions $h_0, h_1, \ldots, h_k$, such that

$$f = h_0,\ g = h_k, \quad \text{and } h_i = r_i \cdot h_{i-1}, \forall i = \overline{1, k}.$$

If both $f \preceq g$ and $g \preceq f$, we say that f and g are associated distributions, and write $f \sim g$.

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Remark

Obviously, $\preceq$ is an order relation in $\mathcal{D}(G)$, and $\sim$ an equivalence relation on $\mathcal{D}(G)$.

Also, if $f, g \in \mathcal{D}(G)$ are such that $f \preceq g$ or $f \sim g$, then $w(f) = w(g)$.

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Problem

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If $f \in \mathcal{D}(G)$ is a distribution on the finite graph G, determine the equivalence class $[f]_\sim$ and its cardinal $|[f]_\sim|$.

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The problem can be treated on each of the connected components of G separately.
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The case of nondirected graphs

Remark

If $G = (V, E)$ is a finite nondirected graph, the relations $\preceq$ and $\sim$ coincide.

Proposition

Let $G = (V, E)$ be a finite nondirected graph, and $x, y \in V$ two vertices, connected by a path of length 2. If $f, g \in \mathcal{D}(G)$ are distributions of equal weight, such that f and g coincide on $V \setminus \{x, y\}$ and $f(x) - g(x) = 1$, then $f \sim g$.

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Bipartite graphs

Remark

A graph $G = (V, E)$ is bipartite if and only if it contains no cycle of odd length. In this case a partition of its vertices can be obtained, starting from an arbitrary vertex a, by taking $A = \{x \in V | \text{there is a path of even length from } a \text{ to } x\}$ and $B = V \setminus A$.

Proposition

Let $G = (V = A \cup B, E)$ be a finite connected, nondirected, bipartite graph, and $f, g \in \mathcal{D}(G)$. Then $f \sim g$ if and only if $w_f(A) = w_g(A)$ and $w_f(B) = w_g(B)$, or $w_f(A) = w_g(B)$ and $w_f(B) = w_g(A)$.

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Bipartite graphs, Cardinal of an equivalence class of distributions

Corollary

Let $G = (V = A \cup B, E)$ *be a finite connected, nondirected, bipartite graph, and* $f \in \mathcal{D}(G)$*. Let* $a = w_f(A)$ *and* $b = w_f(B)$*. Then*

$$|[f]_\sim| = \left(\begin{array}{c} a + |A| - 1 \\ a \end{array} \right) \cdot \left(\begin{array}{c} b + |B| - 1 \\ b \end{array} \right) +$$

$$+ \left(\begin{array}{c} a + |B| - 1 \\ a \end{array} \right) \cdot \left(\begin{array}{c} b + |A| - 1 \\ b \end{array} \right).$$

Nonbipartite graphs

Proposition

Let $G = (V, E)$ be a finite connected nondirected graph, which is not bipartite, and $f, g \in \mathcal{D}(G)$. Then $f \sim g$ if and only if $w(f) = w(g)$.

Corollary

Let $G = (V, E)$ be a finite connected nondirected graph, which is not bipartite, and $f \in \mathcal{D}(G)$. Then

$$|[f]_{\sim}| = \binom{w(f) + |V| - 1}{w(f)}.$$

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The case of directed graphs

Proposition

Let $G = (V, E)$ be a finite directed graph, and $f, g \in \mathcal{D}(G)$. If G is a cycle of length n, then there is a cyclic permutation $\pi \in S_V$, such that, if $f \sim g$, then $f(x) = g(\pi^k(x)), \forall x \in V$, for some $k \in \{1, 2, \ldots, n\}$.

distribution equivalence and strongly connected components

Proposition

Let $G = (V, E)$ be a finite directed graph, and $f, g \in \mathcal{D}(G)$. If $f \sim g$, then for any strongly connected component $K \subseteq V$ of the graph G, the equality $w_f(K) = w_g(K)$ holds.

Corollary

Let $G = (V, E)$ be a finite directed graph, and $f \in \mathcal{D}(G)$. Then the following inequality holds:

$$|[f]_\sim \leq \prod_{K-\text{strongly connected component}} \binom{w_f(K) + |K| - 1}{w_f(K)}.$$

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Thank you!