A representation theorem for reduced Rickart rings

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1 Motivation

2 Introduction
- Reduced Rickart rings
- Sussman rings

3 Fixing Sussman's mistake
- The mistake
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- Finding a convenient subdirect representation

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Outline

History

- An Sussman ring is a certain subdirect product of domains.
- They were introduced in [Sussman 1958].
- Reduced Rickart rings – also known as PP-rings – are similar to them.

How similar are they?

- Is any Sussman ring a reduced Rickart ring?
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- Any commutative reduced Rickart ring should be isomorphic to a Sussman ring
  (according to a lemma by Sussman)
- Task: Generalize Sussman's proof to non-commutative case.
- But there is a mistake in his proof.

New question

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- Every reduced Rickart ring is isomorphic to a Sussman ring.
- A ring is a reduced Rickart ring iff it is isomorphic to a Sussman ring.

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Reduced rings and Rickart rings

Definition

A ring is called **reduced** iff it has no non-zero nilpotent elements.

Definition

A unitary ring R is called a **Rickart ring** iff for every $a \in R$ there are idempotents $e, f \in R$ such that, for all $x \in R$,

- $ax = 0$ iff $ex = x$ (right Rickart),
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Proposition

In a reduced ring, the idempotents e and f from the definition are unique and coincide.

Definition

For every a in a reduced Rickart ring R, let a' be the unique idempotent such that

$$ax = 0 \iff a'x = x$$

for all x. The operation $'$ is called focal operation.

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Examples of reduced Rickart rings

Examples

- $\mathbb{Z}$
- Any Boolean ring
- $\mathbb{Z}_{pq}$ for prime numbers $p \neq q$
- The ring of bounded linear operators on a Hilbert space.

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- Algebraic versions of rings of bounded linear operators on Hilbert spaces:
  - Baer rings [Kaplansky 1955]
  - Rickart rings [Maeda 1960]
- Some facts:
  - A reduced Rickart ring ordered by the Abian order forms a semi-Boolean algebra [Janowitz 1976]
  - A Rickart ring is reduced iff it does not contain any subrings isomorphic to $UTM_2(\mathbb{Z})$ or $UTM_2(\mathbb{Z}_p)$ [Guo and Shum, 2006]

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Sussman rings

Definition (Sussman 1958)

- Let S be a subdirect product of domains $(D_i)_{i\in I}$.
- For every element x of the direct product, define the associate idempotent x°:

$$x_i^\circ := \begin{cases} 0_i & \text{, if } x_i = 0_i \\ 1_i & \text{, else} \end{cases}$$

- If for all x,

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Sussman's proposition

The proof implicitly uses the following:

Wrong proposition [Sussman 1958]

S is a Sussman ring if

- S is a subdirect product of domains,
- for all $x \in S$, there exists $x^\circ \in S$ satisfying
  1. x° is idempotent,
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Counterexample

Example

- Consider all pairs $(0, 2n)$ and $(1, 2n+1)$ with $n \in \mathbb{Z}$.
- They form a subring $R \subset \mathbb{Z}_2 \times \mathbb{Z}$.
- R is a subdirect product of $\mathbb{Z}_2$ and $\mathbb{Z}$.
- $R \simeq \mathbb{Z}$.
- We can define an operation * on R such that
  1. x^* is idempotent,
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Can the wrong proposition be replaced?

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A subdirect products of domains S is a Sussman ring if, for all $x \in S$,

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Can the wrong proposition be replaced?

Improved Sussman's proposition

A subdirect products of domains S is a Sussman ring if, for all $x \in S$,

- if $x_i = 0$ for some i, then x is a zero divisor,
- there exists $x^\circ \in S$ satisfying
  1. x° is idempotent,
  2. $xx^\circ = x^\circ x = x$,
  3. $x + 1 - x^\circ$ is not a zero divisor.

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Does the improved proposition work for us?

Question

- Let R be a reduced Rickart ring.
- Does R have a subdirect representation satisfying the improved proposition?

- We know that R can be subdirectly represented by domains. [Andrunakievich and Ryabukhin]
- We found that, for all $x \in R$, there exists $x^\circ \in R$ such that
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What about the new condition?

Question

Does any subdirect representation of R satisfy the new condition?

If $x_i = 0$ for some i, is x a zero divisor?

Completely prime zero divisor ideals

We modified a lemma proved by Andrunakievich and Ryabukhin:

Lemma

In a reduced ring R, for every $x \in R$, there is an ideal J such that

- $x \notin J_x$,
- J_x *is completely prime, i.e.,*

$$ab \in J_x \Longrightarrow (a \in J_x \vee b \in J_x),$$

- J_x *consists only of zero divisors.*

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A particular subdirect representation

Corollary

- *Let R be a reduced ring.*
- *Then R is isomorphic a subdirect product of all quotients R/J_x.*

Lemma

This subdirect representation satisfies the new condition:

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Any reduced Rickart ring is isomorphic to a Sussman ring

Theorem

A right Rickart ring is isomorphic to a Sussman ring iff it is reduced.

Corollary (Representation theorem)

A ring is reduced Rickart iff it can be subdirectly represented as a Sussman ring.

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Outline

Summary

- Every Sussman ring is a reduced Rickart ring.
- Every reduced Rickart ring can be subdirectly represented as a Sussman ring.

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References

[Sussman, 1958] I. Sussman
A generalization of Boolean rings
Mathematische Annalen 136, 1958, 326-338.

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Thank you for your attention

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