Reflection-closed varieties of multisorted algebras and minor identities II

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Minor terms and minor identities

Recall that a **minor term** is a term of height 1, i.e., a term of the form

$$f(y_1, \ldots, y_n),$$

where $f \in \Sigma$, $\mathrm{dec}(f) = (s_1 \ldots s_n, s)$, $y_i \in Y_{s_i}$, that is,

$$f(\sigma(1), \ldots, \sigma(n)),$$

where $\sigma \colon [n] \to Y$ is a map respecting the sorts, i.e., $\sigma(i) \in Y_{s_i}$ for all $i \in [n]$.

We denote this term by f_σ.

The set of all minor terms of sort s of type τ over Y is denoted by $MT^s_\tau(Y)$.

Minor terms and minor identities

A **minor identity** of type τ over Y is a triple

$$(S', f_\sigma, g_\pi),$$

where $S' \subseteq S$,
$f_\sigma, g_\pi \in MT^s_\tau(Y)$ for some $s \in S$
and the input sorts of f_σ and g_π belong to S'.
The set of all minor identities of type τ over Y is denoted by $MID_\tau(Y)$.

$\mathbf{A} = (A, \Sigma^{\mathbf{A}})$ **satisfies** $(S', f_\sigma, g_\pi) \in MID_\tau(Y)$,
in symbols, $\mathbf{A} \models (S', f_\sigma, g_\pi)$,
if for all valuations $\beta \colon Y|_{S'} \to A$, we have $\beta^\#(f_\sigma) = \beta^\#(g_\pi)$.

Note:

$$\beta^\#(f_\sigma) = f^{\mathbf{A}}(\beta(\sigma(1)), \ldots, \beta(\sigma(n))) = f^{\mathbf{A}}(\beta \circ \sigma).$$

Galois connection Mod–mId

The satisfaction relation induces a Galois connection between heterogeneous algebras and minor identities of type τ.

$$\mathrm{mId}_Y\, \mathcal{K} := \{(S', t_1, t_2) \in \mathit{MID}_\tau(Y) \mid \forall \mathbf{A} \in \mathcal{K} \colon \mathbf{A} \models (S', t_1, t_2)\},$$
$$\mathrm{Mod}\, \mathcal{J} := \{\mathbf{A} \in \mathit{Alg}(\tau) \mid \forall (S', t_1, t_2) \in \mathcal{J} \colon \mathbf{A} \models (S', t_1, t_2)\}.$$

$\mathrm{mId}\, \mathcal{K} := \mathrm{mId}_X\, \mathcal{K}$, where X is the standard set of variables.

In the first part of this presentation (given by Reinhard), we saw that the sets $\mathcal{K}$ of algebras satisfying $\mathcal{K} = \mathrm{Mod}\, \mathrm{mId}\, \mathcal{K}$ are exactly the reflection-closed varieties.

We are now going to characterize the minor-equational theories, i.e., the sets $\mathcal{J}$ of minor identities satisfying $\mathcal{J} = \mathrm{mId}\, \mathrm{Mod}\, \mathcal{J}$.

Canonical trivial algebra

The **canonical trivial algebra of type** $\tau = (S, \Sigma, \mathrm{dec})$ is the algebra

$$\mathbf{S} = (\tilde{S}, \Sigma^{\mathbf{S}})$$

where

- $\tilde{S} = (\tilde{S}_s)_{s \in S}$, $\tilde{S}_s := \{s\}$ for every $s \in S$,
- for each $f \in \Sigma_{(w,s)}$, $f^{\mathbf{S}} \colon \{w\} \to \{s\}$ is trivially defined.

Lemma

For any S-sorted algebra $\mathbf{A} = (A, \Sigma^{\mathbf{A}})$ of type τ, $\tilde{S}|_{S_A}$ is a subalgebra of the canonical trivial algebra $\mathbf{S}$ of type τ, i.e., $\langle \tilde{S}|_{S_A} \rangle_{\mathbf{S}} = \tilde{S}|_{S_A}$.

We will write $\langle S' \rangle_{\mathbf{S}}$ to denote the unique set S'' such that $\langle \tilde{S}|_{S'} \rangle_{\mathbf{S}} = \tilde{S}|_{S''}$.

Minor-equational theories

Theorem

Let $\mathcal{J} \subseteq MID_\tau(X)$ be a set of minor identities of type $\tau = (S, \Sigma, \mathrm{dec})$ over X. Then $\mathcal{J} = \mathrm{mId}\,\mathrm{Mod}\,\mathcal{J}$ if and only if $\mathcal{J}$ satisfies the following conditions:

1. *For every $S' \subseteq S$ and $s \in S$, the set*

$$\mathcal{J}_s^{(S')} := \{(f_\sigma, g_\pi) \mid (S', f_\sigma, g_\pi) \in \mathcal{J}, \mathrm{sort}(f) = \mathrm{sort}(g) = s\}$$

   is an equivalence relation on $MT_\tau^s(X_{S'})$.

2. *If $(S', t_1, t_2) \in \mathcal{J}$ and $S' \subseteq S''$ then $(S'', t_1, t_2) \in \mathcal{J}$.*
3. *If $(S', t_1, t_2) \in MID_\tau(X)$ and $(\langle S' \rangle_{\mathbf{S}}, t_1, t_2) \in \mathcal{J}$, then $(S', t_1, t_2) \in \mathcal{J}$.*
4. *$\mathcal{J}$ is minor-closed, i.e., if $(S', f_\sigma, g_\pi) \in \mathcal{J}$, then $(S', f_{\lambda \circ \sigma}, g_{\lambda \circ \pi}) \in \mathcal{J}$ for all $\lambda \colon X|_{S'} \to X|_{S'}$.*

Proof I

Proof idea. "$\Rightarrow$" Easy.

"$\Leftarrow$" Assume $\mathcal{J}$ satisfies conditions 1–4.

For each $\delta = (S', f_\sigma, g_\pi) \in \mathit{MID}_\tau(X) \setminus \mathcal{J}$, we construct an algebra $\mathbf{F}_\delta \in \mathit{Alg}(\tau)$ such that $\mathbf{F}_\delta \models \mathcal{J}$ but $\mathbf{F}_\delta \not\models \delta$.

Taking $\mathcal{K}$ to be the set of all such "separating" algebras, we get $\mathcal{J} = \mathrm{mId}\,\mathcal{K}$.

Proof II

For $\delta = (S', f_\sigma, g_\pi) \in MID_\tau(X) \setminus \mathcal{J}$, the algebra $\mathbf{F}_\delta = (F, \Sigma^{\mathbf{F}_\delta})$ is defined as follows. Let $S'' := \langle S' \rangle_{\mathbf{S}}$ and $q := \mathrm{sort}(f) = \mathrm{sort}(g)$. For $s \in S$, let

$$F_s := \begin{cases} \emptyset, & \text{if } s \in S \setminus S'', \\ X_s, & \text{if } s \in S'' \setminus \{q\}, \\ X_q \cup MT_\tau^q(X_{S''})/\mathcal{J}_q^{(S'')}, & \text{if } s = q. \end{cases}$$

For $d \in \Sigma_{(w,s)}$, the operation $d^{\mathbf{F}_\delta} \colon F_w \to F_s$ is defined as follows:

- If $s \in S \setminus S''$ then $d^{\mathbf{F}_\delta} := \emptyset$.
- If $s \in S'' \setminus \{q\}$, then $d^{\mathbf{F}_\delta}(\alpha) := x_1^s$ for all $\alpha \in F_w$.
- If $s = q$, then $d^{\mathbf{F}_\delta}(\alpha) := [d_{\varphi \circ \alpha}]$ for all $\alpha \in F_w$, where $\varphi \colon F \to X$ is given by $x_i^s \mapsto x_i^s$ for any $x_i^s \in X|_{S''}$ and $[t] \mapsto x_1^q$ for any $[t] \in MT_\tau^q(X_{S''})/\mathcal{J}_q^{(S'')}$.

Condition 1 and the previous Lemma guarantee that the algebra $\mathbf{F}_\delta$ is well defined.

Proof III

To show: $\mathbf{F}_\delta \not\models \delta = (S', f_\sigma, g_\pi)$.

Let $\beta \colon X|_{S'} \to F$, $x \mapsto x$.

$$\beta^\#(f_\sigma) = f^{\mathbf{F}_\delta}(\beta \circ \sigma) = [f_{\varphi \circ \beta \circ \sigma}] = [f_\sigma],$$
$$\beta^\#(g_\pi) = g^{\mathbf{F}_\delta}(\beta \circ \pi) = [g_{\varphi \circ \beta \circ \pi}] = [g_\pi].$$

Since $(S'', f_\sigma, g_\pi) \notin \mathcal{J}$ by condition 3, we have $[f_\sigma] \neq [g_\pi]$.
Thus $\mathbf{F}_\delta \not\models \delta$.

Proof IV

To show: $\mathbf{F}_\delta \models \mathcal{J}$.

Let $(T, d_\rho, d'_{\rho'}) \in \mathcal{J}$.

If sort$(d) \neq q$, then obviously $\mathbf{F}_\delta \models (T, d_\rho, d'_{\rho'})$.

Assume that sort$(d) =$ sort$(d') = q$.
If $T \nsubseteq S'' = S_F$, then $\mathbf{F}_\delta \models (T, d_\rho, d'_{\rho'})$ holds vacuously.

We may thus assume that $T \subseteq S''$.
Let $\beta \colon X|_T \to F$.
By condition 2, we have $(S'', d_\rho, d'_{\rho'}) \in \mathcal{J}$.
By condition 4, we have $(S'', d_{\varphi\circ\beta\circ\rho}, d'_{\varphi\circ\beta\circ\rho'}) \in \mathcal{J}$.
Then

$$\beta^\#(d_\rho) = d^{\mathbf{F}_\delta}(\beta \circ \rho) = [d_{\varphi\circ\beta\circ\rho}] = [d'_{\varphi\circ\beta\circ\rho'}] = d'^{\mathbf{F}_\delta}(\beta \circ \rho') = \beta^\#(d'_{\rho'}).$$

Thus $\mathbf{F}_\delta \models (T, d_\rho, d'_{\rho'})$.
We conclude that $\mathbf{F}_\delta \models \mathcal{J}$. □

How are HSP and RP related?

Every RP-closed class is also HSP-closed.
The converse is not true.

A type $\tau = (S, \Sigma, \mathrm{dec})$ is **non-composable**
if S can be partitioned into two subsets I and O such that
for every $f \in \Sigma$, we have $\mathrm{in}(f) \subseteq I$ and $\mathrm{sort}(f) \in O$.

Example. 2-algebras

How are HSP and RP related?

Theorem

Let $\tau = (S, \Sigma, \text{dec})$ *be a non-composable type, and let* $\mathcal{K} \subseteq Alg(\tau)$ *be an* HSP*-closed class of algebras. Then the following are equivalent:*

1. $\mathcal{K}$ *is* R*-closed.*
2. *For all* $s \in S$, $\mathcal{K} \not\models (S, x_1^s, x_2^s)$.
3. *For all* $s \in S$*, there exists* $\mathbf{A} \in \mathcal{K}$ *such that* $S_A = S$ *and* $|A_s| \geq 2$.

In fact, implications (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) hold for any similarity type τ.
The crucial part is the implication (2) $\Rightarrow$ (1) (or, equivalently, (3) $\Rightarrow$ (1)).

Proposition

Let τ *be a similarity type. If for every* HSP*-closed class* $\mathcal{K} \subseteq Alg(\tau)$*, conditions* (1)–(3) *of the previous Theorem are equivalent, then* τ *is non-composable.*

How are HSP and RP related?

According to the previous Theorem,
the HSP-varieties of a non-composable type τ that are not RP-varieties are the ones satisfying an identity of the form (S, x_1^s, x_2^s) for some $s \in S$.

$\mathbf{A} = (A, \Sigma^{\mathbf{A}})$ satisfies (S, x_1^s, x_2^s) if and only if $\mathbf{A}$ is "trivial in sort s", i.e., A_s is empty or singleton.

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The end

Thank you for your attention.