Rearrangement Problem of Two Dimensional Arrays by Prefix Reversals

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Understanding Groupoid Actions

- It is known that an inverse semigroup represents partial actions
- On the other hand, we do not know so much on concrete examples of partial actions.
- We focus on concrete examples of partial actions and related groupoids (D-class of inverse semigroups)
- We aim to provide concrete groupoid actions and classify them

Group Actions and Groupoid Actions

- A: Group, S: Set
- Action $A \times S \to S$ is defined for every pair $(a, s) \in A \times S$.
- G: Groupoid, S: Set
- Action $G \times S \to S$ is not necessarily defined for every pair $(g, s) \in G \times S$.
- Multiplication of elements a and b of G is not necessarily defined, either

We know examples of group actions in almost all areas of mathematics, but do not know examples of groupoid actions.

Pancake Sorting Problem

$$\begin{pmatrix}5 & 4 & 1 & 3 & 2\end{pmatrix} \Rightarrow$$
$$\begin{pmatrix}2 & 3 & 1 & 4 & 5\end{pmatrix} \Rightarrow$$
$$\begin{pmatrix}3 & 2 & 1 & 4 & 5\end{pmatrix} \Rightarrow$$
$$\begin{pmatrix}1 & 2 & 3 & 4 & 5\end{pmatrix}$$

3 prefix reversals are applied

Pancake Sorting Problem

- The minimum number of prefix reversals to sort a given list was asked by Gates and Papadimitriou (Discrete Mathematics vol 79, 1979)
- An upper bound $\frac{5}{3}n$ for sorting a list of length n was given by them
- A better bound $\frac{18}{11}n$ was given by Heydari and Sudborough (J. Algorithm vol 25, 1997)
- It is shown NP hard by Bulteau, Fertin and Rusu (MFCS 2012)

Burnt Pancake Sorting Problem

- Instead of numbers, upper and lower case lattes are placed on an array.
- Whenever prefix reversal is applied, case is reversed.
- Some permutations on the alphabet $\{a, b, c, d, e, A, B, C, D, E\}$ can be applied, but the others cannot (groupoid action)

$$\begin{pmatrix} e & d & A & c & b \end{pmatrix} \Rightarrow$$
$$\begin{pmatrix} B & C & a & D & E \end{pmatrix} \Rightarrow$$
$$\begin{pmatrix} c & b & a & D & E \end{pmatrix} \Rightarrow$$
$$\begin{pmatrix} A & B & C & D & E \end{pmatrix}$$

Extension of Pancake Sorting Problem

- We consider two dimensional arrays instead of lists
- Sorting by prefix reversals is always possible for lists
- Rearrangement of two dimensional arrays by prefix reversals is not trivial
- A burnt version seems much more complicated

$n \times m$ Array

- (a_{ij}) is an $n \times m$ array

a_{11}	a_{12}	$\cdots$	a_{1m-1}	a_{1m}
a_{21}	a_{22}	$\cdots$	a_{2m-1}	a_{2m}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_{n1}	a_{n2}	$\cdots$	a_{nm-1}	a_{nm}

- where each integer in $\{1, 2, 3, \ldots, nm\}$ appears as a_{ij} exactly once

Standard Array

- $E_{n\times m}$ (*standard array*)

1	2	$\cdots$	m
$m+1$	$m+2$	$\cdots$	$2m$
$2m+1$	$2m+2$	$\cdots$	$3m$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(n-1)m+1$	$\cdots$	$\cdots$	nm

Reverse of Array

- A: $n \times m$ array

a_{11}	a_{12}	$\cdots$	a_{1m-1}	a_{1m}
a_{21}	a_{22}	$\cdots$	a_{2m-1}	a_{2m}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_{n1}	a_{n2}	$\cdots$	a_{nm-1}	a_{nm}

- $R(A)$: Reverse of A

a_{nm}	a_{nm-1}	$\cdots$	a_{n2}	a_{n1}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_{2m}	a_{2m-1}	$\cdots$	a_{22}	a_{21}
a_{1m}	a_{1m-1}	$\cdots$	a_{12}	a_{11}

Prefix Reversals of Arrays

Prefix reverse is an operation on subarray containing a_{11} entry

- Suppose $A = \frac{A_1}{A_2} = A_3|A_4$
- The transformation $\frac{A_1}{A_2} \Rightarrow \frac{R(A_1)}{A_2}$ is called a horizontal prefix reversal
- The transformation $A_3|A_4 \Rightarrow R(A_3)|A_4$ is called a vertical prefix reversal

Examples of Prefix Reversals

- Let A be the array $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.
- $A \Rightarrow \begin{bmatrix} 3 & 2 & 1 \\ 4 & 5 & 6 \end{bmatrix}$ is a horizontal prefix reversal
- $A \Rightarrow \begin{bmatrix} 5 & 4 & 3 \\ 2 & 1 & 6 \end{bmatrix}$ is a vertical prefix reversal

Rearrangement by Prefix Reversals

$$\begin{bmatrix} 3 & 6 & 5 \\ 2 & 4 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 5 & 6 & 3 \\ 2 & 4 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & 2 & 3 \\ 6 & 5 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 5 & 6 \\ 3 & 2 & 4 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} 6 & 5 & 1 \\ 3 & 2 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & 2 & 3 \\ 1 & 5 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Rearrangement of Array

- Let $A = (a_{ij})$ be an $n \times m$ array
- Suppose $\sigma \in S_{nm}$ is a permutation
  - where S_{nm} is the symmetric group
- $(\sigma(a_{ij}))$ is a rearrangement of A by σ

Example

- Let A be the array $\begin{bmatrix} 3 & 6 & 5 \\ 2 & 4 & 1 \end{bmatrix}$
- $A = \begin{bmatrix} \sigma(1) & \sigma(2) & \sigma(3) \\ \sigma(4) & \sigma(5) & \sigma(6) \end{bmatrix} = \sigma(E_{2\times 3})$
- where σ is the permutation (1 3 5 4 2 6)
- A is a rearrangement of $E_{2\times 3}$ by σ

Rearrangement Problem of Arrays by Prefix Reversals

1. Given a permutation σ in S_{nm} (Symmetric group)
2. Can $E_{n\times m}$ be rearranged from the array $\sigma(E_{n\times m})$ by prefix reversals?
3. If so, how many prefix reversals are needed?

Theorem (AY, Reachability Problem 2015)

For σ in S_{nm}

1. If either $n \not\equiv 0 \pmod 4$ or $m \not\equiv 0 \pmod 4$, $E_{n\times m}$ can be rearranged from $\sigma(E_{n\times m})$ by prefix reversals
2. In the case that $n \equiv m \equiv 0 \pmod 4$, $E_{n\times m}$ can be rearranged from $\sigma(E_{n\times m})$ by prefix reversals if and only if σ belongs to A_{nm} (alternating group)

Symmetric Group and Alternating Group

1. Every permutation is a product of transpositions, that is, 2-cycles
2. Alternating group A_n consists of even permutations, that is, product of even number of transpositions
3. A_n is a proper subgroup of S_n (if $n > 1$)

Sketch of Proof

- If either $n \not\equiv 0 \pmod 4$ or $m \not\equiv 0 \pmod 4$ (rearrangeable cases) we show
  - How to realize any transposition
- If $n \equiv m \equiv 0 \pmod 4$ (not-rearrangeable case) we show
  1. Prefix reversals realizes no transposition
  2. Prefix reversals realize a product of two transpositions

Sketch of Proof

We discuss only cases of n

- Rearrangeable cases
  - $n = 2$: We show any transposition is realized
  - $n = 1, 3 (\text{mod} 4)$ (that is, $n = 2k + 1$):
    We exchange $a_{k+1,1}$ and $a_{k+1,2}$
  - $n = 2 (\text{mod} 4)$ (that is, $n = 4k + 2$):
    We exchange $a_{2k+1,1}$ and $a_{2k+2,1}$
- Not-rearrangeable case
  - $n = 4k$ and $m = 4l$:
    We realize a product of any two disjoint transpositions
    (therefore any even permutations)

Elementary Operations: Reverse of Row and Column

We can always reverse any row (and column) by prefix reversals

$$\begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2 & \cdots & \text{m} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \rightarrow \begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \text{m} & \text{m-1} & \cdots & 2 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

Elementary Operations: Twin Transposition along Column

We can realize the following three permutations by prefix reversals

$$\begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{i,k} & \cdots & a_{i,l} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{j,k} & \cdots & a_{j,l} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \to \begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{j,k} & \cdots & a_{j,l} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{i,k} & \cdots & a_{i,l} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

Elementary Operations: Twin Transposition along Row

$$\begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{i,k} & \cdots & a_{i,l} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{j,k} & \cdots & a_{j,l} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \rightarrow \begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{i,l} & \cdots & a_{i,k} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{j,l} & \cdots & a_{j,k} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

Elementary Operations: Twin Transposition along Diagonal

$$\begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{i,k} & \cdots & a_{i,l} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{j,k} & \cdots & a_{j,l} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \to \begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{j,l} & \cdots & a_{j,k} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{i,l} & \cdots & a_{i,k} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

Case $n = 2k + 1$

We exchange $a_{k+1,1}$ and $a_{k+1,2}$

$$\begin{bmatrix} 1 & 2 & 3 & \cdots \\ 4 & 5 & 6 & \cdots \\ \textcircled{7} & \textcircled{8} & 9 & \cdots \\ 10 & 11 & 12 & \cdots \\ 13 & 14 & 15 & \cdots \end{bmatrix} \to \begin{bmatrix} 14 & 13 & 3 & \cdots \\ 4 & 5 & 6 & \cdots \\ \textcircled{7} & \textcircled{8} & 9 & \cdots \\ 10 & 11 & 12 & \cdots \\ 2 & 1 & 15 & \cdots \end{bmatrix} \to$$

$$\begin{bmatrix} 14 & 13 & 3 & \cdots \\ 11 & 10 & 6 & \cdots \\ \textcircled{7} & \textcircled{8} & 9 & \cdots \\ 5 & 4 & 12 & \cdots \\ 2 & 1 & 15 & \cdots \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & \cdots \\ 4 & 5 & 6 & \cdots \\ \textcircled{8} & \textcircled{7} & 9 & \cdots \\ 10 & 11 & 12 & \cdots \\ 13 & 14 & 15 & \cdots \end{bmatrix}$$

Case $n = 4k + 2$

We exchange $a_{2k+1,1}$ and $a_{2k+2,1}$ positions

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ \textcircled{9} & 10 & 11 & 12 \\ \textcircled{13} & 14 & 15 & 16 \\ 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 \end{bmatrix} \rightarrow \begin{bmatrix} 21 & 2 & 3 & 4 \\ 17 & 6 & 7 & 8 \\ \textcircled{13} & 10 & 11 & 12 \\ \textcircled{9} & 14 & 15 & 16 \\ 5 & 18 & 19 & 20 \\ 1 & 22 & 23 & 24 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 21 & 2 & 3 & 4 \\ 8 & 7 & 6 & 17 \\ \textcircled{13} & 10 & 11 & 12 \\ \textcircled{9} & 14 & 15 & 16 \\ 20 & 19 & 18 & 5 \\ 1 & 22 & 23 & 24 \end{bmatrix} \rightarrow \begin{bmatrix} 21 & 2 & 6 & 17 \\ 8 & 7 & 3 & 4 \\ \textcircled{13} & 10 & 11 & 12 \\ \textcircled{9} & 14 & 15 & 16 \\ 20 & 19 & 23 & 24 \\ 1 & 22 & 18 & 5 \end{bmatrix} \rightarrow$$

Case $n = 4k + 2$

$$\begin{bmatrix} \color{blue}{1} & 2 & 6 & \color{blue}{5} \\ 8 & 7 & 3 & 4 \\ \textcircled{13} & 10 & 11 & 12 \\ \textcircled{9} & 14 & 15 & 16 \\ 20 & 19 & 23 & 24 \\ \color{red}{21} & 22 & 18 & \color{red}{17} \end{bmatrix} \rightarrow \begin{bmatrix} \color{blue}{1} & 2 & 3 & 4 \\ 8 & 7 & 6 & \color{blue}{5} \\ \textcircled{13} & 10 & 11 & 12 \\ \textcircled{9} & 14 & 15 & 16 \\ 20 & 19 & 18 & \color{red}{17} \\ \color{red}{21} & 22 & 23 & 24 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} \color{blue}{1} & 2 & 3 & 4 \\ \color{blue}{5} & 6 & 7 & 8 \\ \textcircled{13} & 10 & 11 & 12 \\ \textcircled{9} & 14 & 15 & 16 \\ \color{red}{17} & 18 & 19 & 20 \\ \color{red}{21} & 22 & 23 & 24 \end{bmatrix}$$

Arbitrary Transposition: $n = 4k + 1$: Exchange of $a_{i,p}$ and $a_{j,q}$

$$\begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i,1} & \cdots & \cdots & a_{i,p} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{k+1,1} & a_{k+1,2} & \cdots & a_{k+1,p} & \cdots & a_{k+1,q} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{j,2} & \cdots & \cdots & \cdots & a_{j,q} \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{k+1,p} & \cdots & \cdots & a_{k+1,1} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i,p} & a_{j,q} & \cdots & a_{i,1} & \cdots & a_{j,2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{k+1,q} & \cdots & \cdots & \cdots & a_{k+1,2} \end{bmatrix} \rightarrow$$

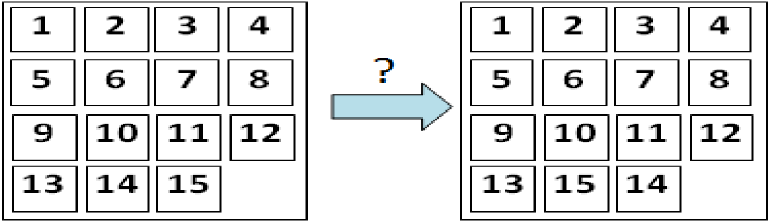
Transposition of Entries in Arbitrary Positions

$$\begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{k+1,p} & \cdots & \cdots & a_{k+1,1} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{j,q} & a_{i,p} & \cdots & a_{i,1} & \cdots & a_{j,2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{k+1,q} & \cdots & \cdots & \cdots & a_{k+1,2} \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i,1} & \cdots & \cdots & a_{j,q} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{k+1,1} & a_{k+1,2} & \cdots & a_{k+1,p} & \cdots & a_{k+1,q} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{j,2} & \cdots & \cdots & \cdots & a_{i,p} \end{bmatrix}$$

Basic Idea: Impossibility of 14-15 puzzle

We use an argument similar to 14-15 Puzzle



Case $n = 4k, m = 4$

A is an array

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,2h} & a_{1,2h+1} & \cdots & a_{1,4h-1} & a_{1,4h} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,2h} & a_{2,2h+1} & \cdots & a_{2,4h-1} & a_{2,4h} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{p-1,1} & a_{p-1,2} & \cdots & a_{p-1,2h} & a_{p-1,2h+1} & \cdots & a_{p-1,4h-1} & a_{p-1,4h} \\ a_{p,1} & a_{p,2} & \cdots & a_{p,2h} & a_{p,2h+1} & \cdots & a_{p,4h-1} & a_{p,4h} \end{bmatrix}$$

Case $n = 4k, m = 4$

$R(A)$ is the array

$$\begin{bmatrix} a_{p,4h} & a_{p,4h-1} & \cdots & a_{p,2h+1} & a_{p,2h} & \cdots & a_{p,2} & a_{p,1} \\ a_{p-1,4h} & a_{p-1,4h-1} & \cdots & a_{p-1,2h+1} & a_{p-1,2h} & \cdots & a_{p-1,2} & a_{p-1,1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{2,4h} & a_{2,4h-1} & \cdots & a_{2,2h+1} & a_{2,2h} & \cdots & a_{2,2} & a_{2,1} \\ a_{1,4h} & a_{1,4h-1} & \cdots & a_{1,2h+1} & a_{1,2h} & \cdots & a_{1,2} & a_{1,1} \end{bmatrix}$$

A product of even number of transpositions $(a_{w,x}, a_{y,z})$ where $1 \leq w, y \leq p$, $1 \leq x \leq 2h$, and $2h + 1 \leq z \leq 4h$ satisfying $w + y = p + 1$ and $x + z = 4h + 1$.

Case $n = 4k$ and $m = 4$

- Prefix reversals generate only even permutations
- An odd permutation σ cannot be generated by prefix reversals
- $E_{n \times m}$ is reachable from $\sigma(E_{n \times m})$ only when σ is even
- (New question) Is every even permutation generated by prefix reversals?

Rearrangeablity of $\sigma(E_{4h\times 4k})$ for Even Permutation σ

How to realize a product of transpositions $(ab)(cd)$

$$\begin{bmatrix} \cdots & a & \cdots & \cdots & \cdots & y & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & b & \cdots & \cdots \\ \cdots & \cdots & c & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & x & \cdots & \cdots & \cdots & d & \cdots \end{bmatrix} \to$$

$$\begin{bmatrix} \cdots & a & \cdots & \cdots & \cdots & b & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & y & \cdots & \cdots \\ \cdots & \cdots & x & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & c & \cdots & \cdots & \cdots & d & \cdots \end{bmatrix} \to$$

Rearrangeability of $\sigma(E_{4h\times 4k})$ for Even Permutation σ

$$\begin{bmatrix} \cdots & b & \cdots & \cdots & \cdots & a & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & y & \cdots & \cdots \\ \cdots & \cdots & x & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & d & \cdots & \cdots & \cdots & c & \cdots \end{bmatrix} \to$$

$$\begin{bmatrix} \cdots & b & \cdots & \cdots & \cdots & y & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & a & \cdots & \cdots \\ \cdots & \cdots & d & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & x & \cdots & \cdots & \cdots & c & \cdots \end{bmatrix}$$

Burnt Version of Rearrangement Problem

1. Let X be an alphabet of upper case letters and $|X| = nm$
2. Let x be an alphabet of lower case letters and $|x| = nm$
3. Given a partial one-one mapping $\sigma \in Sym(X \cup x)$ (Symmetric inverse semigroup on $X \cup x$) such that one of upper case or lower case letter appears exactly once in σ
4. Is $E_{n\times m}$ rearrangeable from the array $\sigma(E_{n\times m})$ by prefix reversals when $\sigma(E_{n\times m})$ is defined.
   - $E_{n\times m}$ is the standard array with upper case letters

Burnt Rearrangement Problem

$$\begin{bmatrix} c & f & e \\ b & D & A \end{bmatrix} \Rightarrow \begin{bmatrix} E & F & C \\ b & D & A \end{bmatrix} \Rightarrow \begin{bmatrix} d & B & C \\ f & e & A \end{bmatrix} \Rightarrow \begin{bmatrix} a & E & F \\ c & b & D \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} f & e & A \\ c & b & D \end{bmatrix} \Rightarrow \begin{bmatrix} d & B & C \\ a & E & F \end{bmatrix} \Rightarrow \begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix}$$

Theorem (AY, R.Kase, S.Fazekas, 2017)

Let n and m be positive integers.

- Suppose either n or m is odd. Given σ in $Sym(2nm)$ where $\sigma(E_{n\times m})$ is defined, the standard array $E_{n\times m}$ is rearrangeable from the array $\sigma(E_{n\times m})$ by prefix reversals.

- Suppose both n and m are even. Given σ in $Sym(2nm)$ where $\sigma(E_{n\times m})$ is defined, the standard array $E_{n\times m}$ is not necessarily rearrangeable from the array $\sigma(E_{n\times m})$ by prefix reversals.

Future Work (Characterization of groupoids)

1. Which partial isomorphism is rearrangeable for arrays of size $2n \times 2m$ and $4n \times 4m$, respectively?
2. Characterize groupoids of such partial isomorphisms

Future Work (Generalization)

1. Generalization: Letters are colored by blue, red, green (or *n* colors in general)
2. Red letter is changed to blue
3. Blue letter is changed to green
4. Green letter is changed to red
5. $\begin{bmatrix} c & f & e \\ b & d & a \end{bmatrix}$ is rearrangeable to $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$
6. Which partial isomorphism can be rearrangeable?