

Clones of small posets

Ádám Kunos

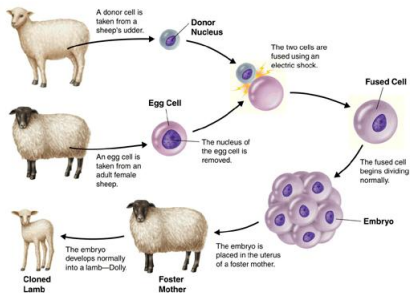
with coauthors Miklós Maróti and László Zádori

University of Szeged

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Clones that normal people have in mind



Intro to our clones

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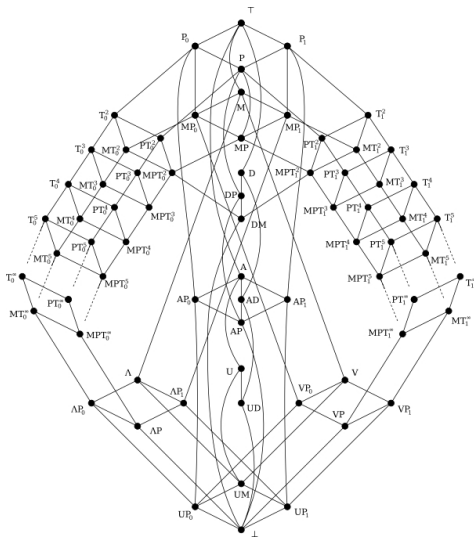
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Clones on a two element set





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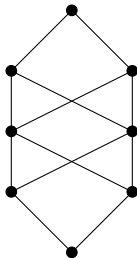
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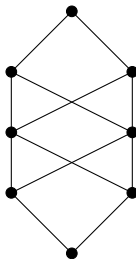
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The first problematic bounded poset has turned out to be:

Nonfinitely generated maximal clones found

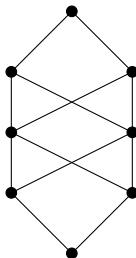


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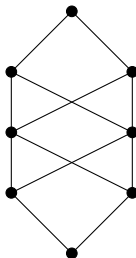
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No one has found nonfinitely generated maximal clones since, though one may conjecture that there are a lot of them.

Main objective, difficulties and our approach

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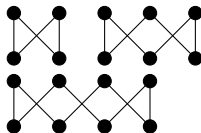


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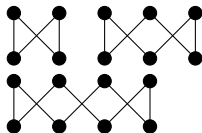


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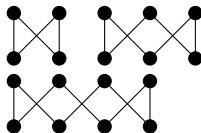
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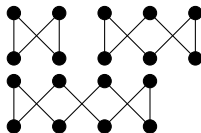
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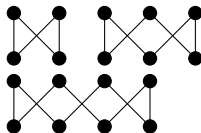
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We described the critical relations of the crowns.

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Proposition

Let P be an arbitrary finite poset and $\alpha \subseteq P^n$. Then α is invariant if and only if there exists a finite poset Q and $(x_1, \dots, x_n) \in Q^n$ for which

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How does the proposition work here?

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We see that there is a connection between critical relations and obstacles: to every critical relation we can assign an obstacle.

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Critical relations of crowns

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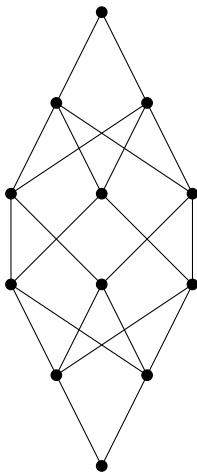
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Theorem

The critical relations of C_{2k} are:

- *the unary \emptyset relation,*
- *the binary critical relations: we can get them with our proposition s. t. Q is a “small” fence and $(x, y) \in Q$ is the pair of the endpoints*
- *the n -ary critical relations, $n \geq 2$: for all large range tuples $\bar{a} \in C_{2k}^n$, the relations $R_{\bar{a}}$.*

Locked 6-crown: $1 + 2 + C_6 + 2 + 1$



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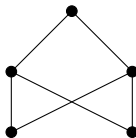
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We felt this might be possible...

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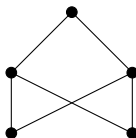
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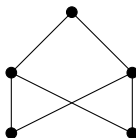
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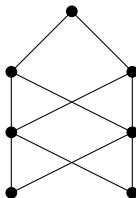
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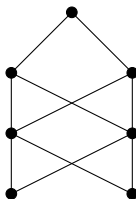
So our observation does not extend to half-bounded posets.

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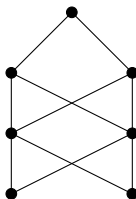
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We also know that its idempotent clone and reduced idempotent clone coincide. QUESTION. Is this clone finitely generated?

Thank you!