Units in quasigroups

Aleksandar Krapež
(with J. Fempl–Madjarević and V. A. Shcherbacov)

Mathematical Institute of the SASA
Belgrade, Serbia

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Dedication

Dedicated to B. Šešelja,

colleague and friend

Quasigroups

Quasigroups: the first definition

A *quasigroup* is a groupoid $(Q; \cdot)$ such that linear equations:

$$a \cdot x = b \qquad\qquad y \cdot a = b$$

are uniquely solvable for all $a, b \in Q$.

Quasigroups

Quasigroups: the second definition

Quasigroups are algebras $(Q; \cdot, /, \backslash)$ satisfying:

$$\begin{aligned} xy/y &= x & \qquad x\backslash xy &= y \\ (x/y)y &= x & \qquad x(x\backslash y) &= y \end{aligned}$$

Quasigroups

Sometimes they are called

- equasigroups
- primitive quasigroups
- 3–quasigroups

A class of all 3–quasigroups is a variety.

Quasigroups

The easiest way to 'understand' quasigroups is:

Quasigroups are 'not necessarily associative' groups. **(WRONG!)**

Groups are associative quasigroups. **(right)**

Quasigroups

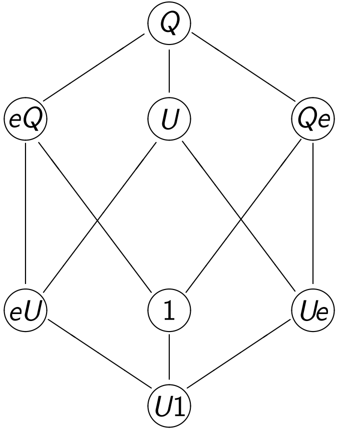
Another 'deficiency' of quasigroups
is that they do not necessarily have a *unit*.

unit = our name for *neutral element* or *identity element*.

Units

unit	symbol	id 1	id 3
none	(Q)	$x = x$	$x = x$
left	(eQ)	$ex = x$	$x/x = y/y$
right	(Qe)	$xe = x$	$x\backslash x = y\backslash y$
middle	(U)	$xx = e$	$xx = yy$
ℓ + r	(1)	$ex = x, xe = x$	$x/x = y\backslash y$
ℓ + m	(eU)	$ex = x, xx = e$	$x/x = yy$
r + m	(Ue)	$xe = x, xx = e$	$x\backslash x = yy$
ℓ + r + m	(U1)	ex = x, xe = x, xx = e	$x/x = y\backslash y = zz$

Units



Belousov's problem

Problem (Belousov (1969))

How to recognize identities which force quasigroups satisfying them to be loops?

Examples

Associativity:

$$x \cdot yz = xy \cdot z$$

which defines groups.

Four Moufang identities:

$$z(x \cdot zy) = (zx \cdot z)y$$

$$x(z \cdot yz) = (xz \cdot y)z$$

$$zx \cdot yz = (z \cdot xy)z$$

$$zx \cdot yz = z(xy \cdot z)$$

which define Moufang loops (Kunen (1996), using Prover).

Another problem

Problem (Krapež, Shcherbacov)

How to recognize identities which force quasigroups satisfying them to have {left, right, middle} unit?

Examples

Left Bol identity:

$$x(y \cdot xz) = (x \cdot yx)z$$

implies existence of a right unit.

Right Bol identity:

$$x(yz \cdot y) = (xy \cdot z)y$$

implies existence of a left unit.

Weak associativity

J. D. H. Smith (2007) proved that the identity of weak associativity:

$$x(y/y) \cdot z = x \cdot (y/y)z$$

implies (1).

Identites similar to weak associativity

Theorem (Krapež, Shcherbacov)

- Identity $x(y\backslash y) \cdot z = x \cdot (y\backslash y)z$ implies (1).

- Identity $(x\backslash yy)\backslash z = x\backslash(yy\backslash z)$ implies (eU).
- Identity $(x\backslash(y/y))\backslash z = x\backslash((y/y)\backslash z)$ implies (eU).

- Identity $(x/yy)/z = x/(yy/z)$ implies (Ue).
- Identity $(x/(y\backslash y))/z = x/((y\backslash y)/z)$ implies (Ue).

Weak associativity – general form

Theorem (Krapež)

Let $(Q;A)$ be cancellative groupoid, $(Q;B)$ groupoid, $(Q;C)$ right quasigroup and $(Q;D)$ left quasigroup. If they satisfy the identity:

$$xA((yBy)Cz) = (xD^{-1}(yDy))Az \qquad \text{(WA)}$$

then there are $e, i \in Q$ such that:

- *e is the middle unit for B,*
- *e is the left unit for C,*
- *i is the middle unit for D.*

The converse also holds.

Parastrophes

Operations $\cdot, *, /, \backslash, /\!\!/, \backslash\!\!\backslash$ are *parastrophes* of $\cdot$ (and of each other):

$$\begin{array}{llllll} x \cdot y = z & \text{iff} & x \backslash z = y & \text{iff} & z / y = x & \text{iff} \\ y * x = z & \text{iff} & z \backslash\!\!\backslash x = y & \text{iff} & y /\!\!/ z = x & \end{array}$$

There are six parastrophes for any quasigroup.

Weak associativity for parastrophes

We are interested in (WA) when A, B, C, D are parastrophes of a quasigroup $\cdot$. Then:

- All four operations are quasigroups;
- The units e and i are equal;
- $e = i$ is some kind of unit for $\cdot$.

On the contrary,
the choice of A does not impose any restriction on $\cdot$.

Partitions of Π

Let us define two partitions on $\Pi = \{\cdot, *, /, \backslash, /\!\!/, \backslash\!\!\backslash\}$:

First: $L = \{/, /\!\!/\}, R = \{\backslash, \backslash\!\!\backslash\}, M = \{\cdot, *\}$

Second: $\ell = \{\cdot, \backslash\}, r = \{*, /\!\!/\}, m = \{/, \backslash\!\!\backslash\}$.

Classification of *WA*–identities

Theorem (Krapež)

(WA) $\Leftrightarrow$ *(eQ) iff* $L\ell L$

iff $A \in \Pi, B \in L, C \in \ell, D \in L.$

There are 48 *WA*–identities equivalent to (eQ).

Example

Let A be $\backslash$, B be $/\!\!/$, C be $\cdot$ and D be $/$.
Then D^{-1} is $\backslash\!\!\backslash$ and the appropriate WA–identity is:

$$x\backslash((y/\!\!/y)\cdot z) = (x\backslash\!\!\backslash(y/y))\backslash z$$

Translating to the language of quasigroups, we get that all quasigroups satisfying:

$$x\backslash((y/y)\cdot z) = ((y/y)\backslash x)\backslash z$$

have left unit.

Classification of *WA*–identities

Analogously:

There are 48 *WA*–identities equivalent to (Qe).

There are 48 *WA*–identities equivalent to (U).

There are 288 *WA*–identities equivalent to (1).

There are 288 *WA*–identities equivalent to (eU).

There are 288 *WA*–identities equivalent to (Ue).

There are 288 *WA*–identities equivalent to (U1).

Closed identities

Definition

Let $s = t$ be a quasigroup identity.
A subterm u of s (t) is *closed* if

- u is a variable, or
- for every variable x from $s = t$,
  u either contains all appearances of x or none.

Identity $s = t$ is *closed* if every subterm u of s (t) is *closed*.

Examples

1. Trivial identity $x = x$ is closed.
2. The identity $xy = xy$ is trivial, but not closed.
3. Collapsing identities $x = y$ and $x = e$ are closed.
4. The identity $x = y \cdot yy$ is collapsing but not closed.

Basic terms

A quasigroup term b is *basic* if:

- b is a variable (b is *linear*); or
- $b = x \circ x$ for some variable x and some $\circ \in \{\cdot, /, \backslash\}$ (b is *quadratic*).

Representation

Theorem (Fempl–Madjarević, Krapež)

Let $s = t$ be nontrivial closed quasigroup identity. Then there are linear terms $L_1[x_1, \ldots, x_k], L_2[x_{k+1}, \ldots, x_n]$ such that
$s = L_1[b_1, \ldots, b_k]$
$t = L_2[b_{k+1}, \ldots, b_n]$
where b_i $(1 \leq i \leq n)$ are basic terms and

$$s = t \Leftrightarrow \bigwedge_{i=1}^{n} (b_i = e).$$

If at least one of b_i is linear, then $s = t$ is collapsing.
Otherwise, $s = t$ is equivalent to the one of:
(eQ), (Qe), (U), (1), (eU), (Ue), (U1).

Translations

Definition

For a given quasigroup $(Q;\cdot,/,\backslash)$, *translations* are:

$$
\begin{aligned}
L_a(x) &= a\cdot x & \quad L_a^{-1}(x) &= a\backslash x\\
R_a(x) &= x\cdot a & \quad R_a^{-1}(x) &= x/a\\
P_a(x) &= x\backslash a & \quad P_a^{-1}(x) &= a/x
\end{aligned}
$$

It is convenient to have numerical values for translations:

	ε	L	R	P	P^{-1}	R^{-1}	L^{-1}
value	0	1	2	3	4	5	6

Definition

We shorten:

$$D(x, y) = \gamma^{-1} A(\alpha x, \beta y)$$

to:

$$D = A(\alpha, \beta, \gamma).$$

Belousov's derivatives

$A_a = A(L_a, \varepsilon, L_a)$ (Right derivative)

${}_aA = A(\varepsilon, R_a, R_a)$ (Left derivative)

$\underset{a}{A} = A(P_a, P_a, \varepsilon)$ (Middle derivative)

Derivatives

$D = A(\alpha, \beta, \gamma)$ is *derivative* if:

- α, β, γ are bijections
- one of them is ε.

$D_a = A(\alpha, \beta, \gamma)$ is *inner derivative (relative to $a \in Q$)* if:

- the one of α, β, γ is ε
- the other two are translations (by a).

Problem

Problem

$D_a = A$ implies the existence of an autotopy of A.
Does it imply the existence of some unit?

Numbers for identities

For every identity $D_a = A$ we have a three digit number pqr (where one of p, q, r is 0 (which stands for ε) and the other two are numbers between 1 and 6 (and they stand for appropriate translations by a). If one of p, q, r is n it means that it stands for all numbers from 1 to 6.

For example, the identity $A_a = A$ is represented by the number 101.

Results

Theorem (Krapež, Shcherbacov)

pqr $\Rightarrow$ (*eQ*) *for*
$pqr \in \{n01, n06, n10, n60, 102, 105, 130, 140, 602, 605, 630, 640\}$.

Results

Analogously:

$pqr \Rightarrow (Qe)$ for
$pqr \in \{0n2, 0n5, 2n0, 5n0, 021, 026, 051, 056, 320, 350, 420, 450\}$.

$pqr \Rightarrow (U)$ for
$pqr \in \{03n, 04n, 30n, 40n, 013, 014, 063, 064, 203, 204, 503, 504\}$.

$pqr \Rightarrow (1)$ for $pqr \in \{210, 260, 510, 560\}$.

$pqr \Rightarrow (eU)$ for $pqr \in \{301, 306, 401, 406\}$.

$pqr \Rightarrow (Ue)$ for $pqr \in \{032, 035, 042, 045\}$.

$pqr \Rightarrow (U1)$ for no pqr.