Higher Δ and Supernilpotence for Congruence Modular Varieties

Andrew Moorhead

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- For an algebra $\mathbb{A}$ and an integer $k \in \mathbb{N}_{\geq 2}$, the higher commutator is a map

$$[\ ,\dots,\ ] : \mathsf{Con}(\mathbb{A})^k \to \mathsf{Con}(\mathbb{A}),$$

  where $[\theta_0, \dots, \theta_{k-1}]$ is defined to be the least congruence satisfying a certain **centrality condition**, which is abbreviated $C(\theta_0, \dots, \theta_{k-2}, \theta_{k-1}; \delta)$.

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$$M(\alpha, \beta) = \left\{ \begin{bmatrix} t(\mathbf{a}_0, \mathbf{b}_1) & t(\mathbf{b}_0, \mathbf{b}_1) \\ t(\mathbf{a}_0, \mathbf{a}_1) & t(\mathbf{b}_0, \mathbf{a}_1) \end{bmatrix} : t \in \text{Pol}(\mathbb{A}), \mathbf{a}_0 \equiv_\alpha \mathbf{b}_0, \mathbf{a}_1 \equiv_\beta \mathbf{b}_1 \right\}$$

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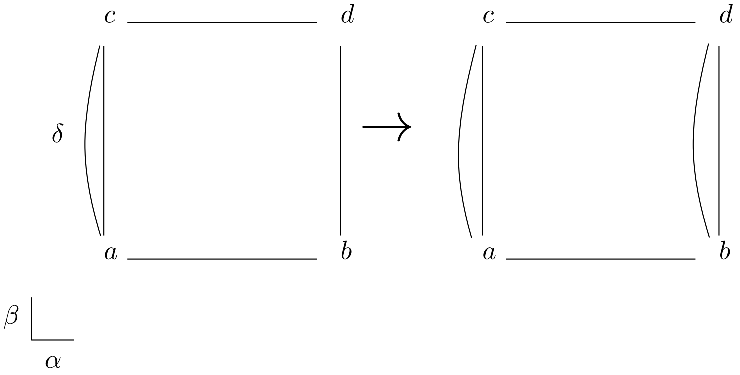
- $M(\alpha, \beta)$ is a subalgebra of $\prod_{f \in 2^2} \mathbb{A}$ with generators

$$\left\{ \begin{bmatrix} x & y \\ x & y \end{bmatrix} : x \equiv_\alpha y \right\} \bigcup \left\{ \begin{bmatrix} y & y \\ x & x \end{bmatrix} : x \equiv_\beta y \right\}$$

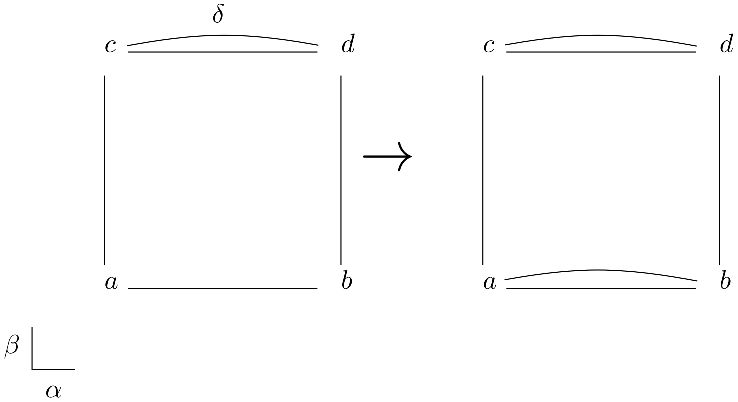
Centrality

For $\delta \in \mathrm{Con}(\mathbb{A})$ we have that α **centralizes** β **modulo** δ if the implication



holds for all (α, β)-matrices. This condition is abbreviated $C(\alpha, \beta; \delta)$.

Similarly, we have that β **centralizes** α **modulo** δ if the implication



holds for all (α, β)-matrices. This condition is abbreviated $C(\beta, \alpha; \delta)$.

Centrality

- The binary commutator is defined to be

$$[\alpha, \beta] = \bigwedge \{\delta : C(\alpha, \beta; \delta)\}$$

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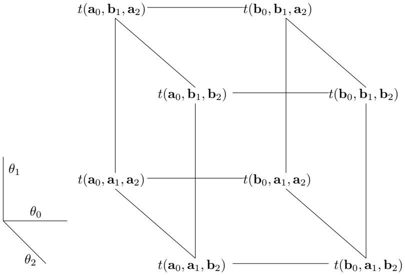
- For congruences $\theta_0, \theta_1, \theta_2$ of $\mathbb{A}$ set $M(\theta_0, \theta_1, \theta_2)$ to be the collection of cubes



for $t \in \mathrm{Pol}(\mathbb{A})$.

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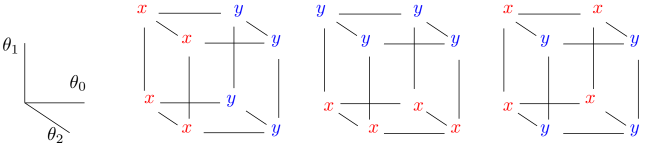

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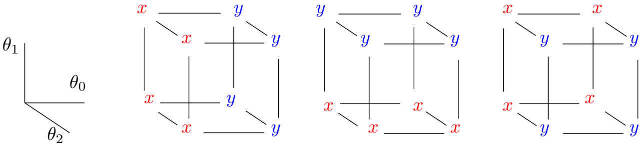

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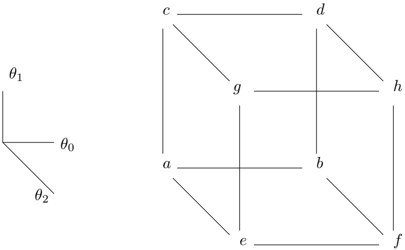

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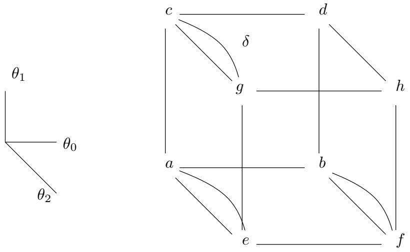

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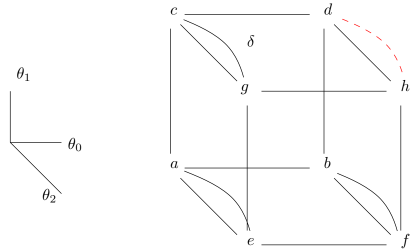

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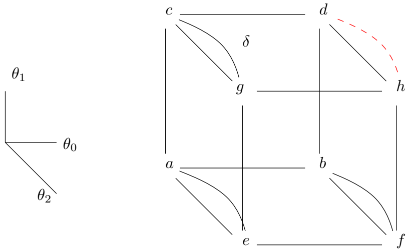
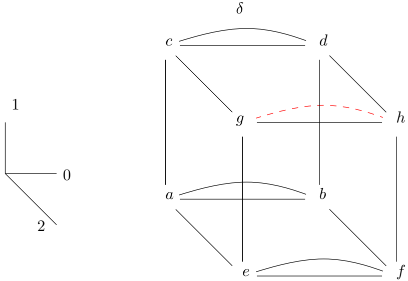


- This condition is abbreviated $C(\theta_0, \theta_1, \theta_2; \delta)$.

Centrality

- Here is a picture of $C(\theta_1, \theta_2, \theta_0; \delta)$:



Matrices

- For congruences $\theta_0, \theta_1, \theta_2$ we set

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- Higher centrality and the commutator for arity ≥ 4 are similarly defined.

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Also,

7. The modular higher commutator is equivalently defined with a two term condition.

Supernilpotence

Definition

Let $\mathbb{A}$ be an algebra. A congruence α of $\mathbb{A}$ is said to be *k*-**step supernilpotent** if

$$\underbrace{[\alpha, \ldots, \alpha]}_{k+1} = 0$$

Commutator Bracketing

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Theorem (Aichinger and Mudrinski 2009)

Let $\mathcal{V}$ *be a permutable variety. Take* $\theta_0, \ldots, \theta_{k-1} \in \mathrm{Con}(\mathbb{A})$ *for* $\mathbb{A} \in \mathcal{V}$. *Then*

$$[[\theta_0, \ldots, \theta_{i-1}], \theta_i, \ldots, \theta_{k-1}] \leq [\theta_0, \ldots, \theta_{k-1}]$$

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$$[[\theta_0, \ldots, \theta_{i-1}], \theta_i, \ldots, \theta_{k-1}] \leq [\theta_0, \ldots, \theta_{k-1}]$$

Therefore, supernilpotent Mal'cev algebras are nilpotent.

Is supernilpotence generally stronger than nilpotence?

Theorem (Wires, 2017)

Let $\mathbb{A}$ be a finite Taylor algebra. If $\mathbb{A}$ is supernilpotent, then it is a nilpotent Mal'cev algebra.

Is supernilpotence stronger?

Theorem (M.)

Let $\mathcal{V}$ be a modular variety. Take $\theta_0, \theta_1, \theta_2 \in \mathsf{Con}(\mathbb{A})$ for $\mathbb{A} \in \mathcal{V}$. Then

$$[[\theta_0, \theta_1], \theta_2] \leq [\theta_0, \theta_1, \theta_2]$$

To show this, we examine a stronger type of centrality and construct Δ for the ternary commutator.

Transitive Term Condition

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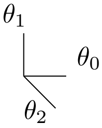
The (ordinary) term condition ranges over $M(\theta_0, \theta_1, \theta_2)$:



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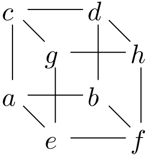

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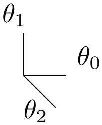

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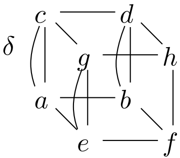

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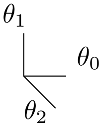

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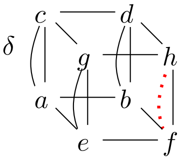

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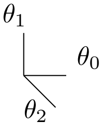

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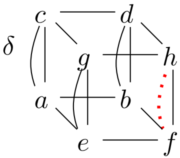

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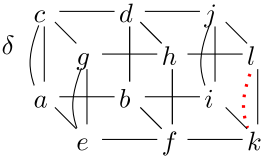
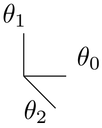
A stronger condition is to demand the following as well:





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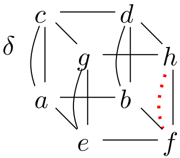

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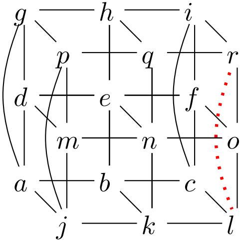
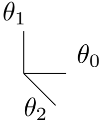
An even stronger condition:





Transitive Term Condition

Definition

Let $\mathbb{A}$ be an algebra, and take $\theta_0, \theta_1, \theta_2 \in \mathrm{Con}(\mathbb{A})$. For $n_0, n_1, n_2 \geq 2$, we call an element

$$c \in \prod_{f \in n_0 \times n_1 \times n_2} \mathbb{A}$$

a $(\theta_0, \theta_1, \theta_2)$-**complex with dimensions** (n_0, n_1, n_2) if the following condition is satisfied:

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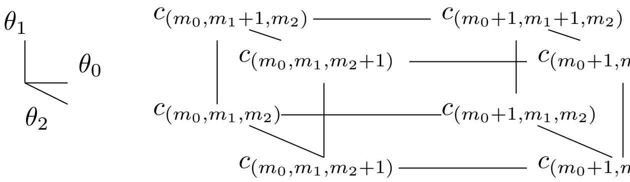
For all $m_0 < n_0$, $m_1 < n_1$, and $m_2 < n_2$, we have that



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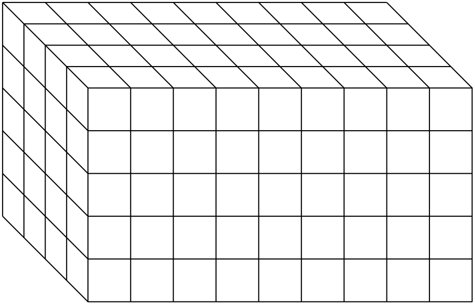
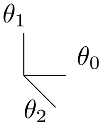
Transitive Term Condition

Take a to be a $(\theta_0, \theta_1, \theta_2)$-complex, with dimensions (n_0, n_1, n_2):





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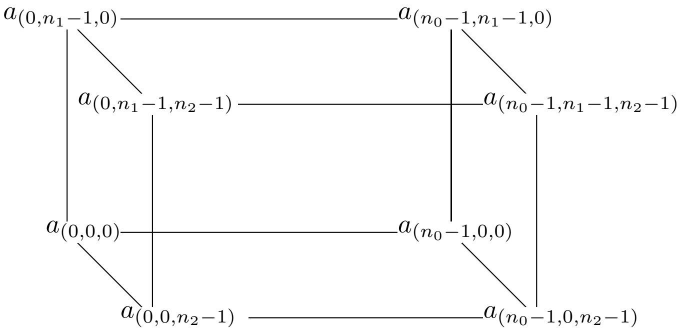
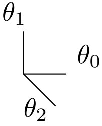


We identify image of the projection onto the corners of a.



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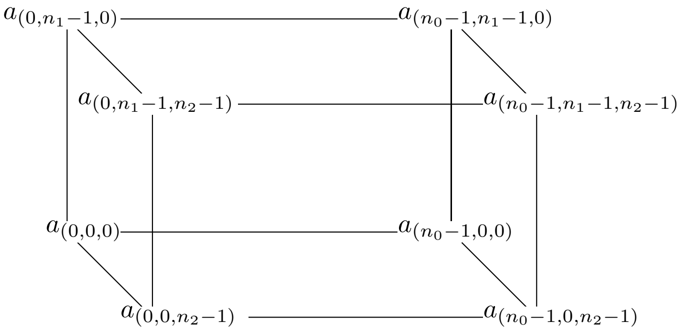

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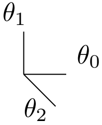
These corners can be mapped onto $\prod_{f\in 2^3} \mathbb{A}$

in the obvious manner.



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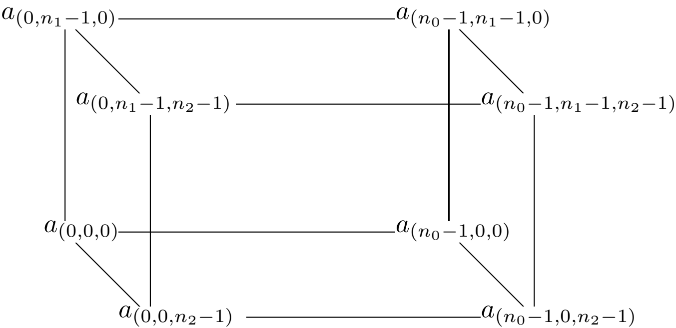

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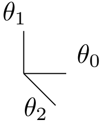

These corners can be mapped onto $\prod_{f \in 2^3} \mathbb{A}$ in the obvious manner.

We call this cube Corner(a).

Transitive Term Condition

Transitive Term Condition

- Denote the collection of all such cubes by Corners$(\theta_0, \theta_1, \theta_2)$.

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- We have that $M(\theta_0, \theta_1, \theta_2) \leq \text{Corners}(\theta_0, \theta_1, \theta_2) \leq \prod_{f \in 2^3} \mathbb{A}$.

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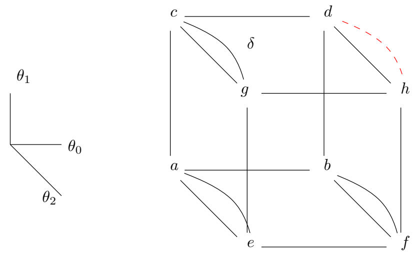
- For $\delta \in \text{Con}(\mathbb{A})$, we say that θ_0, θ_1 **transitively centralize** θ_2 **modulo** δ if the following implication holds for all $(\theta_0, \theta_1, \theta_2)$-corners:



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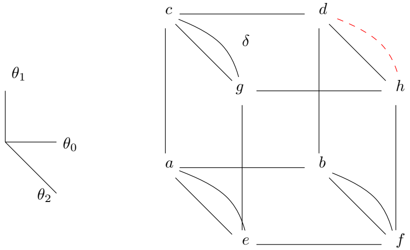

- This condition is abbreviated $C_{tr}(\theta_0, \theta_1, \theta_2; \delta)$

Transitive Term Condition

Theorem (M.)

Let $\mathcal{V}$ be a modular variety, and take $\theta_0, \theta_1, \theta_2, \delta \in \mathrm{Con}(\mathbb{A})$ for $\mathbb{A} \in \mathcal{V}$. The following holds:
$C_{tr}(\theta_0, \theta_1, \theta_2; \delta) \iff C(\theta_0, \theta_1, \theta_2; \delta)$.

Generalizing Δ

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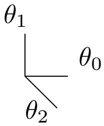
Every $a \in \prod_{f \in 2^3} \mathbb{A}$ can be thought of as a pair of faces in three ways:



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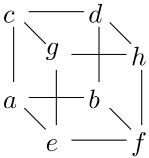

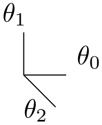

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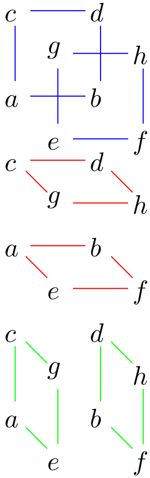

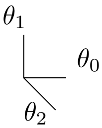

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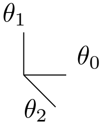
So, a collection of cubes can be considered as a relation on squares in three ways.



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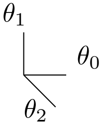
For a collection of cubes X, let X°_i} be the collection of cubes that represent the transitive closure in the i-th direction.



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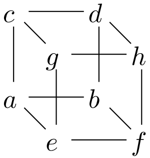
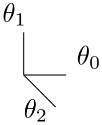
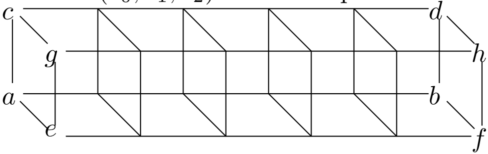
For example,



$\in M(\theta_0, \theta_1, \theta_2)^{\circ_0}$ $\iff$

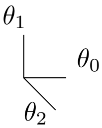


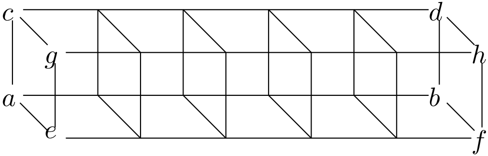
$(\theta_0, \theta_1, \theta_2)$-matrix sequence



Generalizing Δ

Now, $M(\theta_0, \theta_1, \theta_2)^{\circ_0}$ is a collection of cubes. So, it too can be considered as a relation on opposing faces.





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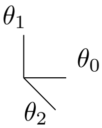
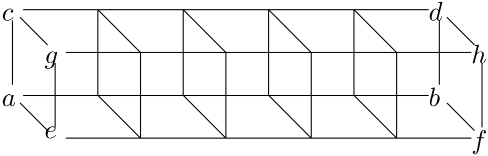
In particular, it can be thought of as a relation on $\Delta_{\theta_0,\theta_1}$ or $\Delta_{\theta_0,\theta_2}$.





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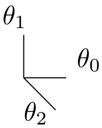

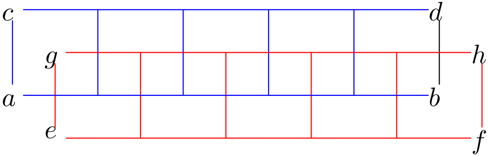

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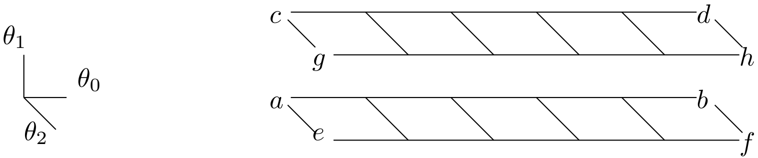

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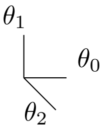
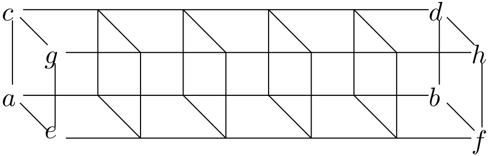
Each of these relations is compatible, reflexive and symmetric, but not necessarily transitive.





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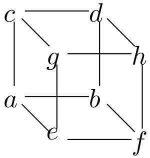
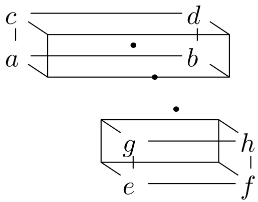
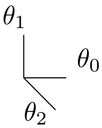
So, take another transitive closure. For example,



$\in M(\theta_0, \theta_1, \theta_2)^{\circ_0,\circ_1} \iff$





Generalizing Δ

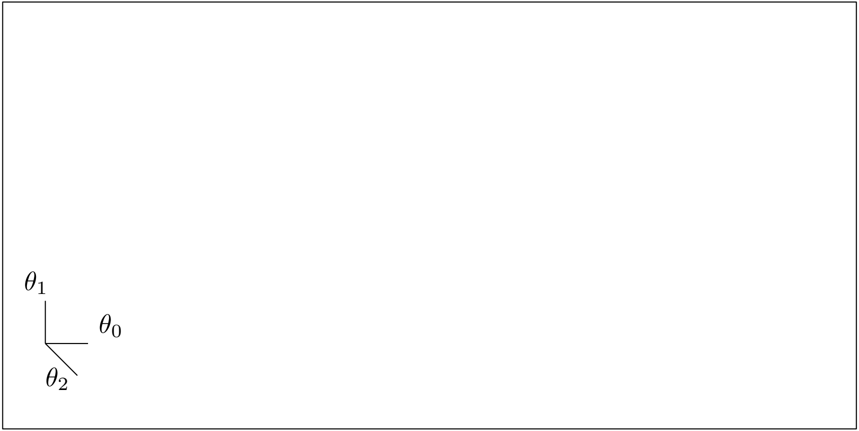
Definition

Let $\mathbb{A}$ be an an algebra. For $\theta_0, \theta_1, \theta_2 \in \mathrm{Con}(\mathbb{A})$ and distinct $i, j, k \in 3$, we define

$$\Delta_{\theta_i, \theta_j, \theta_k} = M(\theta_0, \theta_1, \theta_2)^{\circ_j, \circ_k}$$

Generalizing Δ

By our earlier remarks, $\Delta_{\theta_i,\theta_j,\theta_k}$ can be interpreted as a congruence of $\Delta_{\theta_i,\theta_j}$.



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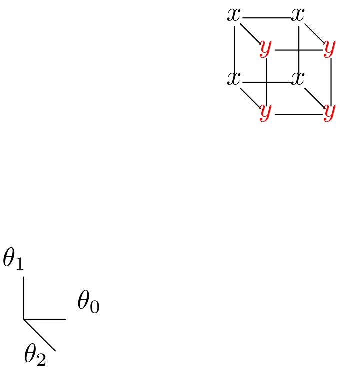

Generators of $\Delta_{\theta_0,\theta_1,\theta_2}$

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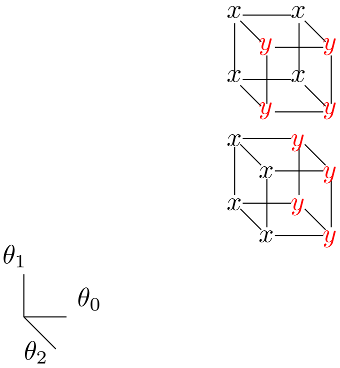

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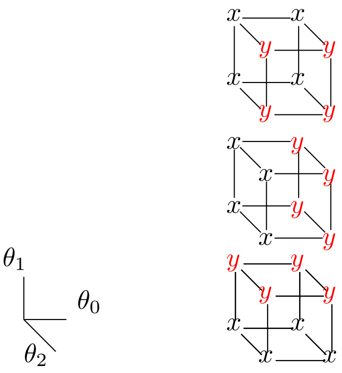

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Generalizing Δ

Proposition (M.)

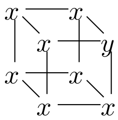
Let $\mathcal{V}$ be a modular variety. Take $\theta_0, \theta_1, \theta_2 \in \mathrm{Con}(\mathbb{A})$ for $\mathbb{A} \in \mathcal{V}$. Then,

$\Delta_{\theta_i,\theta_j,\theta_k} = \Delta_{\theta_{\sigma(i)},\theta_{\sigma(j)},\theta_{\sigma(k)}}$ *for every permutation $\sigma \in S_3$.*

Generalizing Δ

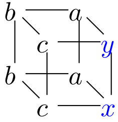
Theorem: Let $\mathcal{V}$ be a modular variety. Take $\theta_0, \theta_1, \theta_2 \in \mathrm{Con}(\mathbb{A})$ for $\mathbb{A} \in \mathcal{V}$. The following are equivalent:

(1) $\langle x, y\rangle \in [\theta_0, \theta_1, \theta_2]$

(2) 

$\in \Delta_{\theta_0,\theta_1,\theta_2}$

There exist elements of $\mathbb{A}$ such that

(3) 

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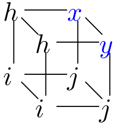
(5) 

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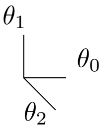
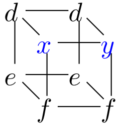


(4) 

$\in \Delta_{\theta_0,\theta_1,\theta_2}$

Using Δ

Theorem (M.)

Let $\mathcal{V}$ be a modular variety. Suppose that C is a ternary congruence lattice operation defined for all congruence lattices across $\mathcal{V}$ that satisfies

1. $C(\theta_0, \theta_1, \theta_2) \leq \theta_0 \wedge \theta_1 \wedge \theta_2$
2. $C(\theta_0, \theta_1, \theta_2) \vee \pi = f^{-1}(C(f(\theta_0 \vee \pi), f(\theta_1 \vee \pi), f(\theta_2 \vee \pi)))$ *for f a surjective homomorphism with kernel π.*

for all $\mathbb{A} \in \mathcal{V}$ and $\theta_0, \theta_1, \theta_0 \operatorname{Con}(\mathbb{A})$.

Then $C(\theta_0, \theta_1, \theta_2) \leq [\theta_0, \theta_1, \theta_2]$ for all $\mathbb{A} \in \mathcal{V}$ and $\theta_0, \theta_1, \theta_0 \in \operatorname{Con}(\mathbb{A})$.

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Corollary

In a modular variety,

$$[[\theta_0, \theta_1], \theta_2] \leq [\theta_0, \theta_1, \theta_2]$$

Using Δ

Theorem (M.)

Let $\mathcal{V}$ be a modular variety. There is a term operation $q(x_0, x_1, x_2, x_3, x_4, x_5, x_6)$ in the clone of $\mathcal{V}$ that satisfies the identities

1. $q(x, x, x, x, y, y, y) \approx y$
2. $q(x, x, y, y, x, x, y) \approx y$

Using Δ

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1. $q(x, x, x, x, y, y, y) \approx y$
2. $q(x, x, y, y, x, x, y) \approx y$

Also,

3. $q^{\mathbb{A}}(x, y, x, y, x, y, x) \equiv_{[\theta,\theta,\theta]} y$
   for all $\mathbb{A} \in \mathcal{V}$, where $\theta = \mathrm{Cg}_{\mathbb{A}}(\langle x, y \rangle)$.