Reconstructing the topology on monoids and polymorphism clones of reducts of the rationals.

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Nada humano me es ajeno

Outline

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1. **Topological Monoids.**
   - Basic open sets
   - Automatic homeomorphicity
2. **Betweenness Relation**
3. Strict circular order
4. Separation relation
5. Polymorphisms Clones

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Presenting joint work with...

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Reconstruction of Topology

Whether we can reconstruct the canonical topology of an endomorphism monoid $\text{End}(\mathbb{A})$ from its underlying abstract monoid structure?

Whether we can reconstruct the canonical topology of a polymorphism clone $\text{Pol}(\mathbb{A})$ from its underlying abstract clone structure?

Automatic homeomorphicity

Meaning that isomorphisms of certain form must necessarily also be homeomorphisms.

Transformation monoids

- For a set A, we denote by $\mathrm{O}_A^{(1)} := A^A$ the set of all unary functions on A and by
$$\mathsf{Tr}(A)$$
the full transformation monoid on A.
- The submonoids
$$M \leq \mathsf{Tr}(A)$$
are transformation monoids on A.

If we equip A with the discrete topology, then $\mathrm{Tr}(A)$ is a product space of A equipped with the **Tychonoff topology**.

Pointwise convergence topology

Let I be an index set. For every finite $J \subseteq I$ and $u : J \to A$:

$$U(J, u) := \{f : I \to A \mid f \restriction_J = u\}.$$

Special case $I = A$, $J = \{a_1^1, \ldots, a_1^m\}$, and we fix m elements $a_0^j = u\left(a_1^j\right) \in A$ for $1 \leq j \leq m$.

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Topology on $\mathrm{Tr}(A)$

A non-empty basic open set is:

$$U(J,u) = \left\{ f\colon A \to A \mid \forall\, 1 \leq j \leq m\colon f\left(a_1^j\right) = u\left(a_1^j\right) = a_0^j \right\}.$$

- Topological monoids are abstract monoids which carry a topology under which the composition is continuous.
- A transformation monoid $M \leq \mathrm{Tr}(A)$ is considered as a topological subspace of $\mathrm{Tr}(A)$.

Given a relational structure $\mathbb{A} = \left(A, (R^{\mathbb{A}})_{\underline{R} \in \Sigma}\right)$, where $R^{\mathbb{A}} \subseteq A^{\mathrm{ar}(\underline{R})}$ for each $\underline{R} \in \Sigma$.

Endomorphism monoids

A function $f \in \mathrm{O}_A^{(1)}$ is called an endomorphism of $\mathbb{A}$ if

$$f : \mathbb{A} \xrightarrow{\text{homo.}} \mathbb{A}.$$

The set of all endomorphisms on $\mathbb{A}$ is denoted by

$$\mathrm{End}\left(\mathbb{A}\right).$$

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Polymorphism

A function $f \in \mathrm{O}_A^{(k)} := A^{A^k}$ is called a polymorphism of $\mathbb{A}$ if

$$f : \mathbb{A}^k \xrightarrow{\text{homo.}} \mathbb{A}.$$

The set of all polymorphisms on $\mathbb{A}$ is denoted by

$$\mathrm{Pol}(\mathbb{A}) = \bigcup_{k \in \mathbb{N}_+} \mathrm{Pol}^{(k)}(\mathbb{A}).$$

Topological closure

Lemma

Let $\mathbf{A} = (A, \mathfrak{P}(A))$, *I a set and consider a subset* $F \subseteq A^I$. *Then we have*

$$\overline{F} = \operatorname{Loc} F,$$

$\operatorname{Loc} F := \{g \in A^I \mid \forall J \subseteq I, J \text{ finite } \exists f \in F\colon f\restriction_J = g\restriction_J\}$.

Using this lemma, $\operatorname{Loc}\operatorname{Pol}(\mathbb{A}) = \operatorname{Pol}(\mathbb{A})$, one can prove

Remark

The submonoid $M \leq \operatorname{Tr}(A)$ is closed $\iff$ $M = \operatorname{End}(\mathbb{A})$ for some relational structure $\mathbb{A}$ with domain A.

Automatic homeomorphicity

Definition (M.Bodirsky, M.Pinsker, A.Pongrácz (2014))

A **closed** monoid $M \leq \mathrm{Tr}\,(A)$ has reconstruction $:\iff$ for every other closed monoid $M' \leq \mathrm{Tr}\,(B)$, if there exists a monoid isomorphism between M and M', then there also exists a monoid isomorphism between M and M' which is a homeomorphism.

Definition (M.Bodirsky, M.Pinsker, A.Pongrácz)

A **closed** monoid $M \leq \mathrm{Tr}\,(A)$ has automatic homeomorphicity $:\iff$ every monoid isomorphism from M to a closed $M' \leq \mathrm{Tr}\,(B)$ is a homeomorphism.

The motivation

For studying such properties is that in the case where G is a closed symmetric group on Ω,
G has **the small index property [SIP]** $\implies$ it has *automatic homeomorphicity*.

SIP

Says that any subgroup of G of index $< 2^{\aleph_0}$ contains the pointwise stabilizer of a finite set.

We investigate the automatic homeomorphicity of the endomorphism monoids of reducts of the rationals, which are:

- Betweenness relation
- Circular order relation
- Separation relation

A Betweenness relation *betw* on $\mathbb{Q}$ is a ternary relation defined by:

$$(\alpha, \beta, \gamma) \in betw \iff (\beta \leq \alpha \leq \gamma) \vee (\gamma \leq \alpha \leq \beta)$$

1. $\mathrm{Aut}\,(\mathbb{Q}, betw) = \langle \mathrm{Aut}\,(\mathbb{Q}, \leq)\,, g\rangle$, where $g(x) = -x$
2. $|\mathrm{Aut}\,(\mathbb{Q}, betw) : \mathrm{Aut}\,(\mathbb{Q}, \leq)| = 2$
3. $\mathrm{Aut}\,(\mathbb{Q}, betw)$ is 2-transitive.

In order to prove automatic homeomorphicity of $\mathrm{Emb}(\mathbb{Q}, betw)$ we use

Lemma (BPP 2014)

Let M be a closed monoid of $Tr(\Omega)$ and $G \leq M$ be the group of its invertible elements. If

1. *G has automatic homeomorphicity (SIP)*
2. *$\overline{G} = M$*
3. *Any injective endomorphism ξ of M which fixes G pointwise is equal to the identity.*

Then, M has automatic homeomorphicity.

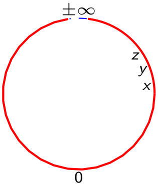
1. We use the fact $|\text{Aut}(\mathbb{Q}, betw) : \text{Aut}(\mathbb{Q}, \leq)| = 2$ and that $\text{Aut}(\mathbb{Q}, \leq)$ has the S.I.P. to prove $\text{Aut}(\mathbb{Q}, betw)$ has S.I.P. (automatic homeomorphicity).
2. We prove that

$$\overline{\text{Aut}(\mathbb{Q}, betw)} = \text{Emb}(\mathbb{Q}, betw) = \text{End}(\mathbb{Q}, strictbetw)$$

3. To prove that if $\xi : \text{Emb}(\mathbb{Q}, betw) \to \text{Emb}(\mathbb{Q}, betw)$ is an injective endomorphism, which fixes every element in G, then ξ fixes every element in $\text{Emb}(\mathbb{Q}, betw)$.

The circular ordering relation on $\mathbb{Q}$

If we twist the (strict) linear order on $\mathbb{Q}$ around the two ends we obtain a (strict) circular order.



A (strict) circular order on $\mathbb{Q}$ is a ternary relation defined by:

$$(x, y, z) \in circ \iff (x < y < z) \vee (y < z < x) \vee (z < x < y)$$

We demonstrate automatic homeomorphicity for its monoid of embeddings.

1. We use the fact $|\mathrm{Aut}\,(\mathbb{Q}, circ) : \mathrm{Aut}\,(\mathbb{Q}, <)| = \aleph_0$ and that $\mathrm{Aut}\,(\mathbb{Q}, <)$ has the S.I.P. to prove $\mathrm{Aut}\,(\mathbb{Q}, circ)$ has S.I.P. (automatic homeomorphicity).
2. We prove that

$$\overline{\mathrm{Aut}\,(\mathbb{Q}, circ)} = \mathrm{Emb}\,(\mathbb{Q}, circ) = \mathrm{End}\,(\mathbb{Q}, circ)$$

3. To prove that if $\xi : \mathrm{Emb}\,(\mathbb{Q}, circ) \to \mathrm{Emb}\,(\mathbb{Q}, circ)$ is an injective endomorphism, which fixes every element in G, then ξ fixes every element in $\mathrm{Emb}\,(\mathbb{Q}, circ)$.

Our remaining task is to show that:

If $\xi(g) = g$ for all $g \in G$, then $\xi(f) = f$ for all $f \in \mathrm{Emb}(\mathbb{Q}, circ)$.

We work with a special subset Γ of $\mathrm{Emb}(\mathbb{Q}, circ)$

1. Note that for any $x, y \in \mathbb{Q}$, we may form the closed interval $[x, y] = \{z : circ(x, z, y)\}$, even if $y < x$ (in which case it actually equals $[x, \infty) \cup (-\infty, y]$ for 'usual' intervals).
2. For any embedding f of $(\mathbb{Q}, circ)$, we define $\sim$ by $x \sim y$ if $[x, y]$ or $[y, x]$ contains at most one point of the image of f
3. If one point, then the interval is *red*; if no point, then it is *blue*.

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2. For any embedding f of $(\mathbb{Q}, circ)$, we define $\sim$ by $x \sim y$ if $[x, y]$ or $[y, x]$ contains at most one point of the image of f
3. If one point, then the interval is *red*; if no point, then it is *blue*.

Γ is taken to be the set of all members f of M all of whose $\sim$-classes are non-empty open intervals, and the red and blue classes form a copy of C_2

The main point is that a key lemma in [BPP] can be applied to members of Γ

Lemma

Any injective endomorphism ξ of $\mathrm{Emb}(\mathbb{Q}, circ)$ which fixes G pointwise also fixes every member of Γ.

The two remaining lemmas used to accomplish this are:

Lemma

If g_1 and g_2 lie in Γ then so does $g_2 g_1$.

Lemma

For any $f \in M$, there is $g \in \Gamma$ such that $gf \in \Gamma$.

Hence

$$\begin{aligned} \xi(g)\xi(f) = \xi(gf) = gf = \xi(g)f \\ \xi(f) = f \qquad\qquad \text{(since } \xi(g) \text{ is left cancellable.)} \end{aligned}$$

There is a group of permutations of a circular order which preserve or reserve it. This gives rise to a quaternary separation relation *sep* defined on $\mathbb{Q}$ by

$$(x, y, z, t) \in sep \iff ((x, y, z) \in circ \wedge (x, t, y) \in circ) \\ \vee \quad ((x, z, y) \in circ \wedge (x, y, t) \in circ)$$

We demostrate how to deduce automatic homeomorphicity for $\mathrm{Emb}(\mathbb{Q}, sep)$

1. We use the fact $|\mathrm{Aut}(\mathbb{Q}, sep) : \mathrm{Aut}(\mathbb{Q}, circ)| = 2$ and that $\mathrm{Aut}(\mathbb{Q}, circ)$ has the S.I.P. to prove $\mathrm{Aut}(\mathbb{Q}, sep)$ has S.I.P. (automatic homeomorphicity).
2. We prove that

$$\overline{\mathrm{Aut}(\mathbb{Q}, sep)} = \mathrm{Emb}(\mathbb{Q}, sep) = \mathrm{End}(\mathbb{Q}, sep)$$

3. To prove that if $\xi : \mathrm{Emb}(\mathbb{Q}, sep) \to \mathrm{Emb}(\mathbb{Q}, sep)$ is an injective endomorphism, which fixes every element in G, then ξ fixes every element in $\mathrm{Emb}(\mathbb{Q}, sep)$. We obtain an injective endomorphism η of $\mathrm{Emb}(\mathbb{Q}, circ)$ which by the result for $\mathrm{Emb}(\mathbb{Q}, circ)$ is the identity ....

To deduce automatic homeomorphicity for

$$\mathrm{Pol}\,(\mathbb{Q}, \leq) \quad \mathrm{Pol}\,(\mathbb{Q}, \mathit{betw}) \quad \mathrm{Pol}\,(\mathbb{Q}, \mathit{circ}) \quad \mathrm{Pol}\,(\mathbb{Q}, \mathit{sep})$$

where

$$\begin{aligned}
(\alpha, \beta, \gamma) \in \mathit{betw} &\iff (\beta \leq \alpha \leq \gamma) \vee (\gamma \leq \alpha \leq \beta) \\
(x, y, z) \in \mathit{circ} &\iff (x \leq y \leq z) \vee (y \leq z \leq x) \vee (z \leq x \leq y) \\
(x, y, z, t) \in \mathit{sep} &\iff ((x, y, z) \in \mathit{circ} \wedge (x, t, y) \in \mathit{circ}) \\
&\quad \vee \quad ((x, z, y) \in \mathit{circ} \wedge (x, y, t) \in \mathit{circ})
\end{aligned}$$

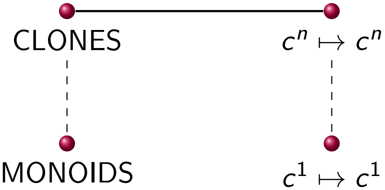
We use

Proposition (BPP 2014)

Let $\mathcal{C}$ be a closed clone with domain C which contains all constant functions on C, and let $\theta : \mathcal{C} \to \mathcal{D}$ be an isomorphism onto a clone $\mathcal{D}$. Then the image of any open subset of $\mathcal{C}$ under θ is open in $\mathcal{D}$.

Lemma (BTV 2016)

Let A and B be sets, P and P' be clones on A and B, respectively, and $\theta\colon P \to P'$ be a clone homomorphism. If for every $b \in B$ there is some unary function $h \in P^{(1)}$ with finite image such that $\theta(h)(b) = b$, then θ is continuous.


CLONES
$c^n \mapsto c^n$
MONOIDS
$c^1 \mapsto c^1$

Acknowledgment

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Thank you :)