Reconstructing the topology on monoids and polymorphism clones of reducts of the rationals.

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Outline

1. **Topological Monoids.**
   - Basic open sets
   - Automatic homeomorphicity

2. Betweenness Relation

3. Strict circular order

4. Separation relation

5. Polymorphisms Clones
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Presenting joint work with...

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Reconstruction of Topology

Whether we can reconstruct the canonical topology of an endomorphism monoid $\text{End}(A)$ from its underlying abstract monoid structure?

Whether we can reconstruct the canonical topology of a polymorphism clone $\text{Pol}(A)$ from its underlying abstract clone structure?
Automatic homeomorphicity

Meaning that isomorphisms of certain form must necessarily also be homeomorphisms.
Transformation monoids

- For a set $A$, we denote by $O_A^{(1)} := A^A$ the set of all unary functions on $A$ and by $\text{Tr}(A)$ the full transformation monoid on $A$.
- The submonoids $M \leq \text{Tr}(A)$ are transformation monoids on $A$. 
If we equip $A$ with the discrete topology, then $\text{Tr}(A)$ is a product space of $A$ equipped with the **Tychonoff topology**.

**Pointwise convergence topology**

Let $I$ be an index set. For every finite $J \subseteq I$ and $u : J \to A$:

$$U(J, u) := \{ f : I \to A \mid f \upharpoonright J = u \}.$$  

Special case $I = A$, $J = \{a_1^1, \ldots, a_1^m\}$, and we fix $m$ elements $a_j^i = u\left(a_1^i\right) \in A$ for $1 \leq j \leq m$.  

If we equip $A$ with the discrete topology, then $\text{Tr}(A)$ is a product space of $A$ equipped with the **Tychonoff topology**.

**Pointwise convergence topology**

Let $I$ be an index set. For every finite $J \subseteq I$ and $u : J \to A$:

$$U(J, u) := \{ f : I \to A \mid f \restriction_J = u \}.$$

Special case $I = A$, $J = \{a_1^1, \ldots, a_1^m\}$, and we fix $m$ elements $a_j^i = u(a_1^i) \in A$ for $1 \leq j \leq m$. 
Topology on $\text{Tr}(A)$

A non-empty basic open set is:

$$U(J, u) = \left\{ f : A \to A \mid \forall 1 \leq j \leq m: f\left(a_1^j\right) = u\left(a_1^j\right) = a_0^j \right\}.$$ 

- **Topological monoids** are abstract monoids which carry a topology under which the composition is continuous.
- A transformation monoid $M \leq \text{Tr}(A)$ is considered as a topological subspace of $\text{Tr}(A)$. 
Given a relational structure $\mathbb{A} = \left( A, \left( R^\mathbb{A} \right)_{R \in \Sigma} \right)$, where $R^\mathbb{A} \subseteq A^{\text{ar}(R)}$ for each $R \in \Sigma$.

**Endomorphism monoids**

A function $f \in O_A^{(1)}$ is called an endomorphism of $\mathbb{A}$ if

$$f : \mathbb{A} \overset{\text{homo.}}{\longrightarrow} \mathbb{A}.$$  

The set of all endomorphisms on $\mathbb{A}$ is denoted by

$$\text{End} \left( \mathbb{A} \right).$$
Given a relational structure $\mathbb{A} = \left( A, (R^\mathbb{A})_{R \in \Sigma} \right)$, where $R^\mathbb{A} \subseteq A^{\text{ar}(R)}$ for each $R \in \Sigma$.

**Polymorphism**

A function $f \in O_A^{(k)} := A^{A^k}$ is called a polymorphism of $\mathbb{A}$ if

$$f : \mathbb{A}^k \xrightarrow{\text{homo.}} \mathbb{A}.$$

The set of all polymorphisms on $\mathbb{A}$ is denoted by

$$\text{Pol} (\mathbb{A}) = \bigcup_{k \in \mathbb{N}_+} \text{Pol}^{(k)} (\mathbb{A}) .$$
Topological closure

Lemma

Let $A = (A, \mathcal{P}(A))$, $I$ a set and consider a subset $F \subseteq A^I$. Then we have

$$
\overline{F} = \text{Loc } F,
$$

where

$$
\text{Loc } F := \{g \in A^I \mid \forall J \subseteq I, J \text{ finite } \exists f \in F : f \upharpoonright J = g \upharpoonright J\}.
$$

Using this lemma, $\text{Loc Pol } (\mathcal{A}) = \text{Pol } (\mathcal{A})$, one can prove

Remark

The submonoid $M \leq \text{Tr } (A)$ is closed $\iff M = \text{End } (\mathcal{A})$ for some relational structure $\mathcal{A}$ with domain $A$. 
Automatic homeomorphicity

Definition (M. Bodirsky, M. Pinsker, A. Pongrácz (2014))

A closed monoid $M \leq \text{Tr}(A)$ has reconstruction $:\iff$ for every other closed monoid $M' \leq \text{Tr}(B)$, if there exists a monoid isomorphism between $M$ and $M'$, then there also exists a monoid isomorphism between $M$ and $M'$ which is a homeomorphism.

Definition (M. Bodirsky, M. Pinsker, A. Pongrácz)

A closed monoid $M \leq \text{Tr}(A)$ has automatic homeomorphicity $:\iff$ every monoid isomorphism from $M$ to a closed $M' \leq \text{Tr}(B)$ is a homeomorphism.
For studying such properties is that in the case where $G$ is a closed symmetric group on $\Omega$, $G$ has the small index property [SIP] $\Rightarrow$ it has *automatic homeomorphism*.

**SIP**

Says that any subgroup of $G$ of index $< 2^{\aleph_0}$ contains the pointwise stabilizer of a finite set.
We investigate the automatic homeomorphicity of the endomorphism monoids of reducts of the rationals, which are:

- Betweenness relation
- Circular order relation
- Separation relation
A **Betweenness relation** betw on \( \mathbb{Q} \) is a ternary relation defined by:

\[
(\alpha, \beta, \gamma) \in \text{betw} \iff (\beta \leq \alpha \leq \gamma) \lor (\gamma \leq \alpha \leq \beta)
\]

1. \( \text{Aut} (\mathbb{Q}, \text{betw}) = \langle \text{Aut} (\mathbb{Q}, \leq), g \rangle \), where \( g(x) = -x \)
2. \( |\text{Aut} (\mathbb{Q}, \text{betw}) : \text{Aut} (\mathbb{Q}, \leq)| = 2 \)
3. \( \text{Aut} (\mathbb{Q}, \text{betw}) \) is 2-transitive.
In order to prove automatic homeomorphicity of \( \text{Emb}(\mathbb{Q}, \text{betw}) \) we use

**Lemma (BPP 2014)**

Let \( M \) be a closed monoid of \( \text{Tr}(\Omega) \) and \( G \leq M \) be the group of its invertible elements. If

1. \( G \) has automatic homeomorphicity (SIP)
2. \( \overline{G} = M \)
3. Any injective endomorphism \( \xi \) of \( M \) which fixes \( G \) pointwise is equal to the identity.

Then, \( M \) has automatic homeomorphicity.
We use the fact $|\text{Aut}(\mathbb{Q}, \text{betw}) : \text{Aut}(\mathbb{Q}, \leq)| = 2$ and that $\text{Aut}(\mathbb{Q}, \leq)$ has the S.I.P. to prove $\text{Aut}(\mathbb{Q}, \text{betw})$ has S.I.P. (automatic homeomorphicity).

We prove that

\[
\text{Aut}(\mathbb{Q}, \text{betw}) = \text{Emb}(\mathbb{Q}, \text{betw}) = \text{End}(\mathbb{Q}, \text{strictbetw})
\]

To prove that if $\xi : \text{Emb}(\mathbb{Q}, \text{betw}) \to \text{Emb}(\mathbb{Q}, \text{betw})$ is an injective endomorphism, which fixes every element in $G$, then $\xi$ fixes every element in $\text{Emb}(\mathbb{Q}, \text{betw})$. 
The circular ordering relation on $\mathbb{Q}$

If we twist the (strict) linear order on $\mathbb{Q}$ around the two ends we obtain a (strict) circular order.

A (strict) circular order on $\mathbb{Q}$ is a ternary relation defined by:

$$(x, y, z) \in \text{circ} \iff (x < y < z) \lor (y < z < x) \lor (z < x < y)$$

We demonstrate automatic homeomorphicity for its monoid of embeddings.
1. We use the fact $|\text{Aut} (\mathbb{Q}, \text{circ}) : \text{Aut} (\mathbb{Q}, <)| = \aleph_0$ and that $\text{Aut} (\mathbb{Q}, <)$ has the S.I.P. to prove $\text{Aut} (\mathbb{Q}, \text{circ})$ has S.I.P. (automatic homeomorphicity).

2. We prove that 

$$\text{Aut} (\mathbb{Q}, \text{circ}) = \text{Emb} (\mathbb{Q}, \text{circ}) = \text{End} (\mathbb{Q}, \text{circ})$$

3. To prove that if $\xi : \text{Emb} (\mathbb{Q}, \text{circ}) \rightarrow \text{Emb} (\mathbb{Q}, \text{circ})$ is an injective endomorphism, which fixes every element in $G$, then $\xi$ fixes every element in $\text{Emb} (\mathbb{Q}, \text{circ})$. 
Our remaining task is to show that:

If $\xi(g) = g$ for all $g \in G$, then $\xi(f) = f$ for all $f \in \text{Emb}(\mathbb{Q}, \text{circ})$.

We work with a special subset $\Gamma$ of $\text{Emb}(\mathbb{Q}, \text{circ})$

1. Note that for any $x, y \in \mathbb{Q}$, we may form the closed interval $[x, y] = \{z : \text{circ}(x, z, y)\}$, even if $y < x$ (in which case it actually equals $[x, \infty) \cup (-\infty, y]$ for ‘usual’ intervals).

2. For any embedding $f$ of $(\mathbb{Q}, \text{circ})$, we define $\sim$ by $x \sim y$ if $[x, y]$ or $[y, x]$ contains at most one point of the image of $f$.

3. If one point, then the interval is red; if no point, then it is blue.
Our remaining task is to show that:

If \( \xi(g) = g \) for all \( g \in G \), then \( \xi(f) = f \) for all \( f \in \text{Emb}(\mathbb{Q}, \text{circ}) \).

We work with a special subset \( \Gamma \) of \( \text{Emb}(\mathbb{Q}, \text{circ}) \):

1. Note that for any \( x, y \in \mathbb{Q} \), we may form the closed interval \( [x, y] = \{ z : \text{circ}(x, z, y) \} \), even if \( y < x \) (in which case it actually equals \( [x, \infty) \cup (-\infty, y] \) for ‘usual’ intervals).

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Our remaining task is to show that:

If $\xi(g) = g$ for all $g \in G$, then $\xi(f) = f$ for all $f \in \text{Emb}(\mathbb{Q}, \circlearrowright)$.

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2. For any embedding $f$ of $(\mathbb{Q}, \circlearrowright)$, we define $\sim$ by $x \sim y$ if $[x, y]$ or $[y, x]$ contains at most one point of the image of $f$.

3. If one point, then the interval is red; if no point, then it is blue.
Γ is taken to be the set of all members \( f \) of \( M \) all of whose \( \sim \)-classes are non-empty open intervals, and the red and blue classes form a copy of \( C_2 \).

The main point is that a key lemma in [BPP] can be applied to members of \( \Gamma \).

**Lemma**

*Any injective endomorphism* \( \xi \) *of* \( \text{Emb}(\mathbb{Q}, \text{circ}) \) *which fixes* \( G \) *pointwise also fixes every member of* \( \Gamma \).
The two remaining lemmas used to accomplish this are:

**Lemma**

*If* $g_1$ *and* $g_2$ *lie in* $\Gamma$ *then so does* $g_2 g_1$.

**Lemma**

*For any* $f \in M$, *there is* $g \in \Gamma$ *such that* $gf \in \Gamma$.

**Hence**

\[
\xi(g)\xi(f) = \xi(gf) = gf = \xi(g)f
\]

\[
\xi(f) = f \quad \text{(since } \xi(g) \text{ is left cancellable.)}
\]
There is a group of permutations of a circular order which preserve or reserve it. This gives rise to a quaternary separation relation \( sep \) defined on \( \mathbb{Q} \) by

\[
(x, y, z, t) \in sep \iff ((x, y, z) \in \text{circ} \land (x, t, y) \in \text{circ})
\land ((x, z, y) \in \text{circ} \land (x, y, t) \in \text{circ})
\]

We demonstrate how to deduce automatic homeomorphicity for \( \text{Emb} (\mathbb{Q}, sep) \)
1. We use the fact $|\text{Aut}(\mathbb{Q}, \text{sep}) : \text{Aut}(\mathbb{Q}, \text{circ})| = 2$ and that $\text{Aut}(\mathbb{Q}, \text{circ})$ has the S.I.P. to prove $\text{Aut}(\mathbb{Q}, \text{sep})$ has S.I.P. (automatic homeomorphismicity).

2. We prove that

$$\text{Aut}(\mathbb{Q}, \text{sep}) = \text{Emb}(\mathbb{Q}, \text{sep}) = \text{End}(\mathbb{Q}, \text{sep})$$

3. To prove that if $\xi : \text{Emb}(\mathbb{Q}, \text{sep}) \rightarrow \text{Emb}(\mathbb{Q}, \text{sep})$ is an injective endomorphism, which fixes every element in $G$, then $\xi$ fixes every element in $\text{Emb}(\mathbb{Q}, \text{sep})$. We obtain an injective endomorphism $\eta$ of $\text{Emb}(\mathbb{Q}, \text{circ})$ which by the result for $\text{Emb}(\mathbb{Q}, \text{circ})$ is the identity ....
To deduce automatic homeomorphicity for

\[
\begin{align*}
\text{Pol} (\mathbb{Q}, \leq) & \quad \text{Pol} (\mathbb{Q}, \text{betw}) & \quad \text{Pol} (\mathbb{Q}, \text{circ}) & \quad \text{Pol} (\mathbb{Q}, \text{sep}) \\
\end{align*}
\]

where

\[(\alpha, \beta, \gamma) \in \text{betw} \iff (\beta \leq \alpha \leq \gamma) \lor (\gamma \leq \alpha \leq \beta)\]

\[(x, y, z) \in \text{circ} \iff (x \leq y \leq z) \lor (y \leq z \leq x) \lor (z \leq x \leq y)\]

\[(x, y, z, t) \in \text{sep} \iff ((x, y, z) \in \text{circ} \land (x, t, y) \in \text{circ}) \lor ((x, z, y) \in \text{circ} \land (x, y, t) \in \text{circ})\]
We use

**Proposition (BPP 2014)**

*Let $\mathcal{C}$ be a closed clone with domain $C$ which contains all constant functions on $C$, and let $\theta : \mathcal{C} \to \mathcal{D}$ be an isomorphism onto a clone $\mathcal{D}$. Then the image of any open subset of $\mathcal{C}$ under $\theta$ is open in $\mathcal{D}$.***

**Lemma (BTV 2016)**

*Let $A$ and $B$ be sets, $P$ and $P'$ be clones on $A$ and $B$, respectively, and $\theta : P \to P'$ be a clone homomorphism. If for every $b \in B$ there is some unary function $h \in P^{(1)}$ with finite image such that $\theta(h)(b) = b$, then $\theta$ is continuous.*
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Thank you :}