TECHNISCHE UNIVERSITÄT WIEN

Centralizing monoids with majority witnesses on four-element domains

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A Galois correspondence

$\mathcal{O}_A^{(n)} = A^{A^n} = \{ f \mid f\colon A^n \longrightarrow A \}$ $\mathcal{O}_A = \bigcup_{n\in\mathbb{N}_+} A^{A^n}$

Commutation of $f \in \mathcal{O}_A^{(m)}$ with $g \in \mathcal{O}_A^{(n)}$

$$g \perp f \;:\Longleftrightarrow\; g \in \operatorname{Hom}\left(\langle A; f\rangle^n ; \langle A; f\rangle\right)$$

Centralizer of $F \subseteq \mathcal{O}_A$

$$F^* = \{ g \in \mathcal{O}_A \mid \forall f \in F\colon g \perp f \} = \bigcup_{n\in\mathbb{N}_+} \operatorname{Hom}\left(\langle A; F\rangle^n ; \langle A; F\rangle\right)$$

(that's a clone)

$$F^{**} = (F^*)^* \supseteq \langle F\rangle_{\mathcal{O}_A} \supseteq F$$

(bicentralizer)

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Unary commuting operations

$F \subseteq \mathcal{O}_A$

$F^{*(1)} = \mathrm{Hom}\,(\langle A; F\rangle\,; \langle A; F\rangle) = \mathrm{End}\,(\langle A; F\rangle).$

unary part of a centralizer = endomorphism monoid of an algebra

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$s \in \mathcal{O}_A^{(1)}$

$s \in F^{*(1)} \iff \forall f \in F\, \forall \mathbf{x} \in A^{\operatorname{ar} f}: s(f(\mathbf{x})) = f(s \circ \mathbf{x})$

Centralizing monoids + observations

For $M \subseteq \mathcal{O}_A^{(1)}$ TFAE

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$|A| = 3 \implies$ minimal majority functions and constants f suffice as witnesses

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The three-element set

Rosenberg, Machida 2010–11: $|A| = 3$

description of all centralizing monoids (192), 10 maximal

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centralizing monoids with majority ops / semiprojs as witnesses

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- partition $\mathcal{O}_A^{(1)}$ into classes C
  (e.g. by $|\operatorname{im} s|$: permutations, 2-el. image, const.)

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  $\rightsquigarrow$ precise description of $\{s\}^* \cap \mathrm{Maj}$
  $\rightsquigarrow$ observation when $\{s_1\}^* \cap \mathrm{Maj} = \{s_2\}^* \cap \mathrm{Maj}$

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- logical inferences betw conditions: e.g. contradiction

Exploiting Goldstern, Machida, Rosenberg

$\sigma := \{ (x, y, z) \in A^3 \mid x \neq y \neq z \neq x \}$ 24 triples with distinct entries

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other tuples, e.g. $s(f(a,a,b)) = s(a) = f(s(a), s(a), s(b))$

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Functions with $|\mathrm{im}\, s| = 2$

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For $|A| = 4\ldots$

$s \in \mathcal{O}_A^{(1)}$ with $\text{im}(s) = \{\alpha, \beta\}$, $\quad |s^{-1}[\{\beta\}]| = 2$

$\text{Maj} \ni f \perp s \iff$

$$\forall (a,b,c) \in \sigma: \quad f(a,b,c) \in \begin{cases} s^{-1}[\{\beta\}] & \text{if } s^{-1}[\{\beta\}] \subseteq \{a,b,c\} \\ s^{-1}[\{\alpha\}] & \text{else.} \end{cases}$$

The four-element set

- $\left|\mathcal{O}_A^{(1)}\right| = 4^4 = 2^8 = 256$ unary operations

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- 4 constants ✓
- $7 \cdot 12 = 84$ $s \in \mathcal{O}_A^{(1)}$ with $|\operatorname{im} s| = 2$
  2 lemmas, 7 cases $\rightsquigarrow$ ✓
- 144 $s \in \mathcal{O}_A^{(1)}$ with $|\operatorname{im} s| = 3$???
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Setting the scene

$f \in \mathrm{Maj}$, $s \in \mathcal{O}_A^{(1)}$, $\mathrm{im}\, s = \{\alpha, \beta, \gamma\}$

$A = \{u, v, x, y\}$

$\bullet$	u	v	x	y
$s(\bullet)$	α	α	β	γ

- $s \triangleright \{\alpha, \beta, \gamma\}$,

$$s\restriction_{\{\alpha,\beta,\gamma\}} \notin \mathrm{Sym}\,\{\alpha, \beta, \gamma\} \iff \{u, v\} \subseteq \{\alpha, \beta, \gamma\}$$

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$$A = \{u, v, x, y\} \quad \begin{array}{c||c|c|c|c} \bullet & u & v & x & y \\ \hline s(\bullet) & \alpha & \alpha & \beta & \gamma \end{array}.$$

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- Case 1: $s{\restriction}_{\{\alpha,\beta,\gamma\}}$ not a permutation
- Case 2: $s{\restriction}_{\{\alpha,\beta,\gamma\}}$ is a permutation

How to enumerate all $f \in \{s\}^*$

$f \in \mathrm{Maj}$, $s \in \mathcal{O}_A^{(1)}$

$s \perp f \iff \forall \mathbf{x} \in \sigma: \quad s(f(\mathbf{x})) = f(s \circ \mathbf{x})$

Construct a graph:

$$\begin{array}{c} \mathbf{x} \mapsto \quad s \circ \mathbf{x} \mapsto \ldots \mapsto \mathbf{y} \\ f(\mathbf{x}) \Leftarrow f(s \circ \mathbf{x}) \Leftarrow \ldots \Leftarrow f(\mathbf{y}) \end{array}$$

where $f(\mathbf{y})$ is given by the majority law,
e.g. $\mathbf{y} = (\alpha, \alpha, \beta)$, so $f(\mathbf{y}) = \alpha$.

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- $f(u, x, y) \in s^{-1}\left[\{f(\alpha,\beta,\gamma)\}\right]$
- $f(v, x, y) \in s^{-1}\left[\{f(\alpha,\beta,\gamma)\}\right]$

Case 2: $s\restriction_{\{\alpha,\beta,\gamma\}} \in \mathrm{Sym}\,\{\alpha,\beta,\gamma\}$

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$\{u,v\} \not\subseteq \{\alpha,\beta,\gamma\}$, assume $v \notin \{\alpha,\beta,\gamma\} \ni u$,

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- $f(v,x,y) \in s^{-1}[\{f(\alpha,\beta,\gamma)\}]$ (det. by values on orbit)

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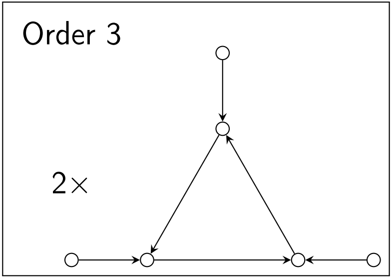
Picture!



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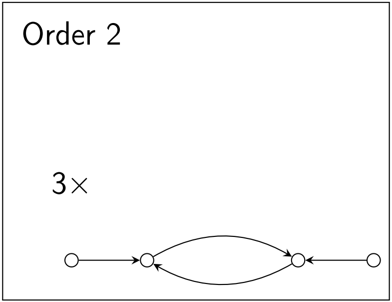

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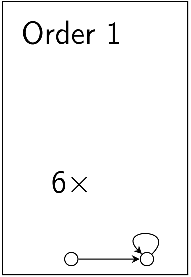

That's already good enough...

For $s \in \mathcal{O}_A^{(1)} \setminus \{\mathrm{id}_A, c_0, c_1, c_2, c_3\}$

- enumerate $\{s\}^* \cap \mathrm{Maj}$

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For $s \in \mathcal{O}_A^{(1)} \setminus \{\mathrm{id}_A, c_0, c_1, c_2, c_3\}$

- enumerate $\{s\}^* \cap \mathrm{Maj}$
- $\forall f \in \{s\}^* \cap \mathrm{Maj}$: compute $\{f\}^{*(1)}$

...since it gives the complete picture

...because

- exhaust all 144 remaining $s \in \mathcal{O}_A^{(1)}$
  + full enumeration of $\{s\}^* \cap \mathrm{Maj} \implies$ full information

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- exhaust all 144 remaining $s \in \mathcal{O}_A^{(1)}$ + full enumeration of $\{s\}^* \cap \mathrm{Maj} \implies$ full information
- $\forall F \subseteq \mathrm{Maj}\colon F' := F^* \cap \mathcal{O}_A^{(1)} = \bigcap_{f \in F} \{f\}^* \cap \mathcal{O}_A^{(1)}$

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  $F' = \{\mathrm{id}_A, c_0, \ldots, c_3\}$
- or F' is an intersection of centralisers already computed
  $\forall f \in F\, \exists s \in \{f\}^* \cap \left(\mathcal{O}_A^{(1)} \setminus \{\mathrm{id}_A, c_0, c_1, c_2, c_3\}\right)\colon$
  $f \in \{s\}^* \cap \mathrm{Maj}$

The result

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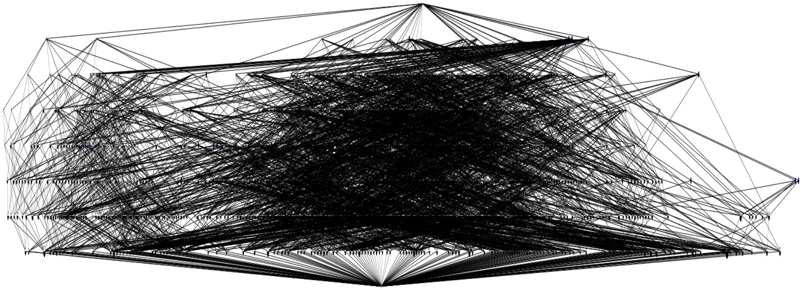
- AAA 93: without 144 s with $|\mathrm{im}\, s| = 3$: $\geq 1\,595$ monoids
- **Final result:** $= 1\,715$ (after getting the computer to programme itself, and some bookkeeping)
- Only $392/4^{24}$ majority op's are needed to differentiate between all the monoids.
- Only $155/256$ unary op's are needed to differentiate between all the monoids.

A glimpse at the lattice. . .

. . . without 144 functions with 3-element image



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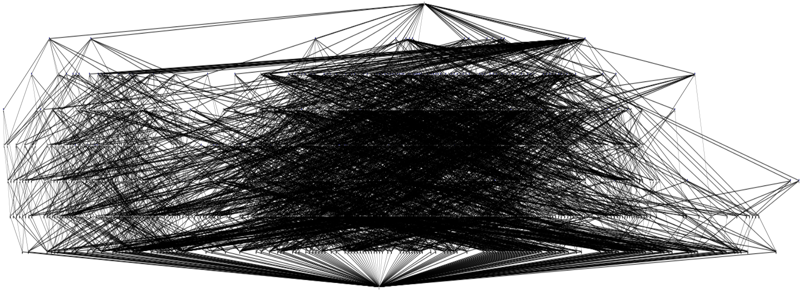

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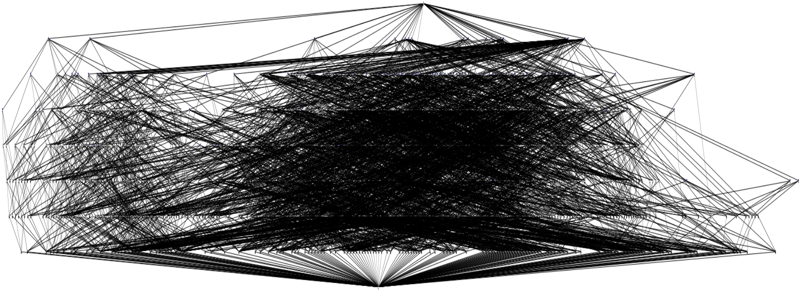

Remarkably small increase for all the additional computations.

Further questions

- Number up to isomorphism?
- Classification of the monoids
- What about semiprojections?
- 5 elements, majority witnesses: feasible?

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Thank you for your attention.