Reconstructing the topology of the elementary self-embedding monoids of countable saturated structures

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(joint work with Maja Pech)

Topology on transformation monoids

- Given a set A, equipped with the discrete topology.
- T_A denotes the monoid of all transformations of A.

Topology on T_A

- for every finite $M \subseteq A$ and for every $h : M \to A$:

$$\Phi_{M,h} := \{f \in T_A \mid f\restriction_M = h\}.$$

- together, all $\Phi_{M,h}$ form the basis of a topology — the topology of pointwise convergence on T_A,
- Composition of functions is continuous.

Topology on transformation monoids

- Every transformation monoid $M \leq T_A$ can be considered as topological subspace of T_A.

Closed permutation groups/transformation monoids

Definition

- $M \leq T_A$ is called closed if it is a closed subspace of T_A.
- $G \leq S_A$ is called closed if it is a closed subspace of S_A.

- $M \leq T_A$ is closed iff $M = \mathrm{End}(\mathbf{A})$, for some structure $\mathbf{A}$ on A.
- $G \leq S_A$ is closed iff $G = \mathrm{Aut}(\mathbf{A})$, for some structure $\mathbf{A}$ on A.
- Usually, permutation groups are not closed if considered as transformation monoids.
- If $G \leq S_A$, then we denote the closure of G in T_A by $\overline{G}$.
- We have $h \in \overline{G}$ iff $\forall M \subseteq A$ finite $\exists g \in G : h\restriction_M = g\restriction_M$.
- Clearly, $\overline{G} \leq T_A$.

Example

- $\overline{\mathrm{Aut}(\mathbb{Q}, <)} = \mathrm{End}(\mathbb{Q}, <)$,
- for countable homogeneous $\mathbf{A}$: $\overline{\mathrm{Aut}(\mathbf{A})} = \mathrm{Emb}(\mathbf{A})$,
- for countable $\aleph_0$-categorical $\mathbf{A}$: $\overline{\mathrm{Aut}(\mathbf{A})} = \mathrm{EEmb}(\mathbf{A})$.

Automatic homeomorphicity

Given:

- a closed permutation group $G \leq S_A$,
- a closed transformation monoid $M \leq S_A$.

Definition (Bodirsky, Pinsker, Pongrácz)

- G has automatic homeomorphicity if every group isomorphism to another closed permutation group on A is a homeomorphism.
- M has automatic homeomorphicity if every monoid isomorphism to another closed transformation monoid on A is a homeomorphism.

Monoids with automatic homeomorphicity:

- $T_{\mathbb{N}}$,
- the monoid of injective functions from $T_{\mathbb{N}}$
- the endomorphism monoid of the Rado graph,
- the endomorphism monoids of $(\mathbb{Q}, <)$ and of $(\mathbb{Q}, \leq)$,
- ...

Formulation of the problem

Given a closed permutation group $G \leq S_A$.

Problem (Bodirsky, Pinsker, Pongrácz 2015)

Suppose that G has automatic homeomorphicity. Does $\overline{G}$ have automatic homeomorphicity?

Criterion by Bodirsky, Pinsker, Pongrácz

Let A be a countable set, and let $G \leq S_A$ be a closed permutation group that has automatic homeomorphicity. If the only injective monoid-endomorphism of $\overline{G}$ that fixes every element of G is the identity on $\overline{G}$, then $\overline{G}$ has automatic homeomorphicity, too.

Some bibliographical facts about the problem

Bodirsky, Pinsker, Pongrácz showed that the criterion applies to the automorphism groups of every countable relational structure **A** with the "joint extension property", such that Aut(**A**) has automatic homeomorphicity and no algebraicity.

Example

- $(\mathbb{N}, =)$,
- the Rado graph,
- the countable random tournament,
- the countable random digraph,
- the countable random uniform hypergraphs.

Behrisch, Truss, Vargas-García showed that the criterion applies to Aut$(\mathbb{Q}, <)$.

Truss and Vargas-García generalized this to the automorphism groups of the first order reducts of $(\mathbb{Q}, <)$.

The main result

Definition

Let $\mathbf{A}$ be an L-structure and let λ be an infinite cardinal. Then $\mathbf{A}$ is called λ-saturated if for every L-structure $\mathbf{B}$ holds:

$$\forall\kappa < \lambda\, \forall\bar{a} \in A^\kappa\, \forall\bar{b} \in B^\kappa : (\mathbf{A}, \bar{a}) \equiv (\mathbf{B}, \bar{b}) \Rightarrow \forall d \in B\, \exists c \in A : (\mathbf{A}, \bar{a}c) \equiv (\mathbf{B}, \bar{b}d).$$

$\mathbf{A}$ is called saturated if it is $|A|$-saturated.

Theorem

Let $\mathbf{A}$ *be a countable structure such that*

- $\mathbf{A}$ *is* ω*-saturated,*
- $\mathrm{Aut}(\mathbf{A})$ *has automatic homeomorphicity,*
- $\mathrm{Aut}(\mathbf{A})$ *has a trivial center.*

Then $\overline{\mathrm{Aut}(\mathbf{A})}$ *has automatic homeomorphicity, too.*

Some examples

The closures of the automorphism groups of the following structures, all have automatic homeomorphicity:

- all the examples by Bodirsky/Pinsker/Pongrácz, Behrisch/Truss/Vargas-García, and Truss/Vargas-García,
- the countable random poset,
- the rational Urysohn space and the rational Urysohn sphere,
- the Henson graphs and the Henson digraphs,
- the countable atomless Boolean algebra,
- the edge-colored random graphs with countably many colors,
- ω-stable, ω-categorical structures whose automorphism groups have a trivial center.

General strategy of the proof

Theorem

Let $\mathbf{A}$ *be a countable saturated structure such that* $\mathrm{Aut}(\mathbf{A})$ *has automatic homeomorphicity, and such that* $\mathrm{Aut}(\mathbf{A})$ *has a trivial center. Then* $\overline{\mathrm{Aut}(\mathbf{A})}$ *has automatic homeomorphicity, too.*

Structure of the proof

- Identify a subset of special elements of $\overline{\mathrm{Aut}(\mathbf{A})}$.
- Show that every injective endomorphism of $\overline{\mathrm{Aut}(\mathbf{A})}$ that fixes $\mathrm{Aut}(\mathbf{A})$ element-wise, fixes each special element.
- Show that for each $\iota \in \overline{\mathrm{Aut}(\mathbf{A})}$ there exist special elements $u, v \in \overline{\mathrm{Aut}(\mathbf{A})}$, such that $u \circ \iota = v$.

- This strategy was used by Behrisch/Truss/Vargas-García for showing automatic homeomorphicity of $\overline{\mathrm{Aut}(\mathbb{Q}, \leq)}$.
- Special elements of $\overline{\mathrm{Aut}(\mathbb{Q}, \leq)}$ are the maximally spread out self-embeddings.

Special self-embeddings

Definition

Let **U** be a structure. A substructure $\mathbf{V} \leq \mathbf{U}$ is called superhomogeneous in **U** if

1. every local isomorphism of **V** extends to an automorphism of **U** whose restriction to V is an automorphism of **V**, and if
2. for all $y \in U \setminus V$ there exists some $\alpha \in \mathrm{Aut}(\mathbf{U})$ that fixes V pointwise such that $\alpha(y) \neq y$.

Example (superhomogeneous substructures:)

- clopen intervals in $(\mathbb{Q}, \leq)$ are superhomogeneous in $(\mathbb{Q}, \leq)$,
- maximally spread out substructures of $(\mathbb{Q}, \leq)$.

Definition

Let **U** be a countable homogeneous structure. A self-embedding ι of **U** is called special if its image is superhomogeneous in **U**.

Encoding special self-embeddings by automorphisms

Definition

Let $G \leq S_A$ be a permutation group and let $M \leq T_A$ be the closure of G in T_A. For every $f \in M$, $x \in A$ we define

$$f^* := \{(\alpha, \beta) \mid \alpha, \beta \in G, \alpha \circ f = f \circ \beta\},$$

$$f^*(x) := \bigcap \{\operatorname{fix}(\alpha) \mid (\alpha, \beta) \in f^*, x \in \operatorname{fix}(\beta)\},$$

$$I(f) := \bigcup_{x \in A} f^*(x).$$

Easy observation:

$$\forall x : f(x) \in f^*(x), \qquad \operatorname{im}(f) \subseteq I(f).$$

The situation for special self-embeddings

Lemma

Given:

- *a countable homogeneous structure* $\mathbf{A}$,
- $f \in \overline{\mathrm{Aut}(\mathbf{A})}$ *with superhomogeneous image,*
- $h \in \mathrm{End}(\overline{\mathrm{Aut}(\mathbf{A})})$ *fixing* $\mathrm{Aut}(\mathbf{A})$ *element-wise.*

Then

1. $I(f) = \mathrm{im}(f)$,
2. $\mathrm{im}(f) = \mathrm{im}(h(f))$.

Proposition

With the notions from above, let $g \in \overline{\mathrm{Aut}(\mathbf{A})}$, *such that* $\mathrm{im}(f) = \mathrm{im}(g)$. *Then*

$$f^* = g^* \iff \exists \zeta \in Z(\mathrm{Aut}(\mathbf{U})) : g = f \circ \zeta.$$

The existence of special self-embeddings

Definition

We call a countable saturated structure smooth if its signature is countable and if its theory admits quantifier-elimination.

Lemma

For every countable saturated structure $\mathbf{A}$ *there exists a smooth countable saturated structure* $\widetilde{\mathbf{A}}$*, such that* $\mathrm{Aut}(\mathbf{A}) = \mathrm{Aut}(\widetilde{\mathbf{A}})$.

Proposition

Let $\mathbf{A}$ *be a smooth countable saturated structure, and let* $\mathbf{B}$ *be a substructure of* $\mathbf{A}$ *isomorphic to* $\mathbf{A}$*. Then there exists an extension* $\widetilde{\mathbf{A}}$ *of* $\mathbf{A}$*, such that* $\widetilde{\mathbf{A}} \cong \mathbf{A}$ *and such that both,* $\mathbf{A}$ *and* $\mathbf{B}$*, are superhomogeneous in* $\widetilde{\mathbf{A}}$*.*

About the proof

Proposition

Let $\mathbf{A}$ *be a smooth countable saturated structure, and let* $\mathbf{B}$ *be a substructure of* $\mathbf{A}$ *isomorphic to* $\mathbf{A}$*. Then there exists an extension* $\widetilde{\mathbf{A}}$ *of* $\mathbf{A}$*, such that* $\widetilde{\mathbf{A}} \cong \mathbf{A}$ *and such that both,* $\mathbf{A}$ *and* $\mathbf{B}$*, are superhomogeneous in* $\widetilde{\mathbf{A}}$*.*

- The construction takes place inside of a $\aleph_1$-big elementary extension $\widehat{\mathbf{A}}$ of $\mathbf{A}$.
- Since $\mathbf{A}$ is ω-saturated, $\operatorname{Age}(\widehat{\mathbf{A}}) = \operatorname{Age}(\mathbf{A})$.
- 5 robots, together, build up a tower of substructures of $\widehat{\mathbf{A}}$ whose union is $\widetilde{\mathbf{A}}$.
- Each robot takes care about one desired property of $\widetilde{\mathbf{A}}$.

The work of the robots

Robot 1 takes care that every local isomorphism of $\mathbf{B}$ extends to an automorphism of $\widetilde{\mathbf{A}}$ whose restriction to B is an automorphism of $\mathbf{B}$.

Robot 2 takes care that every local isomorphism of $\mathbf{A}$ extends to an automorphism of $\widetilde{\mathbf{A}}$ whose restriction to A is an automorphism of $\mathbf{A}$.

Robot 3 takes care that for every $b \in \widetilde{A} \setminus B$ there exists an automorphism of $\widetilde{\mathbf{A}}$ that fixes B pointwise but that does not fix b.

Robot 4 takes care that for every $a \in \widetilde{A} \setminus A$ there exists an automorphism of $\widetilde{\mathbf{A}}$ that fixes A pointwise but that does not fix a.

Robot 5 takes care that $\widetilde{\mathbf{A}}$ is going to be homogeneous.

Robots 3 and 4 make use of the fact that $\mathbf{A}$ is ω-saturated.

Remainder of the proof

Proposition

Let $\mathbf{A}$ *be a smooth countable saturated structure, and let* $\mathbf{B}$ *be a substructure of* $\mathbf{A}$ *isomorphic to* $\mathbf{A}$*. Then there exists an extension* $\widetilde{\mathbf{A}}$ *of* $\mathbf{A}$*, such that* $\widetilde{\mathbf{A}} \cong \mathbf{A}$ *and such that both,* $\mathbf{A}$ *and* $\mathbf{B}$*, are superhomogeneous in* $\widetilde{\mathbf{A}}$*.*

Consequences

- For every $\iota \in \overline{\mathrm{Aut}(\mathbf{A})}$ there exist $u, v \in \overline{\mathrm{Aut}(\mathbf{A})}$ with superhomogeneous image, such that $u \circ \iota = v$.
- The only endomorphism of $\overline{\mathrm{Aut}(\mathbf{A})}$ that fixes $\mathrm{Aut}(\mathbf{A})$ element-wise is the identity.
- By the BPP-criterion, $\overline{\mathrm{Aut}(\mathbf{A})}$ has automatic homeomorphicity.

Definition

A countable $\aleph_0$-categorical structure $\mathbf{A}$ is called *G-finite* if its automorphism group has a smallest closed finite index subgroup.

Theorem (Lascar '89)

Let $\mathbf{A}$ be a countable G-finite, $\aleph_0$-categorical structure, and let $\mathbf{B}$ be another countable structure. Let φ be a monoid isomorphism from $\mathrm{EEmb}(\mathbf{A})$ to $\mathrm{EEmb}(\mathbf{B})$. Then $\varphi\restriction_{\mathrm{Aut}(\mathbf{A})}$ is a topological isomorphism from $\mathrm{Aut}(\mathbf{A})$ to $\mathrm{Aut}(\mathbf{B})$.

Theorem

Let $\mathbf{A}$ be a countable G-finite, $\aleph_0$-categorical structure whose automorphism group has a trivial center, and let $\mathbf{B}$ be another countable structure. Let φ be a monoid isomorphism from $\mathrm{EEmb}(\mathbf{A})$ to $\mathrm{EEmb}(\mathbf{B})$. Then φ is a homeomorphism.

Application 2

- Consider the countable random poset $(\mathbb{P}, <)$.
- Consider $C = \mathrm{Pol}(\mathbb{P}, <)$,
- C is a clone. I.e., it contains all projections and is closed with respect to composition.
- Clone-isomorphisms are bijections between clones that preserve projections and composition.

Theorem

Let $\mathbf{A}$ *be a countable* $\aleph_0$*-categorical structure. Then every clone-isomorphism from* $\mathrm{Pol}(\mathbb{P}, <)$ *to* $\mathrm{Pol}(\mathbf{A})$ *is a homeomorphism.*

If $\mathrm{Aut}(\mathbb{P}, <)$ has automatic homeomorphicity, then the assumption in the theorem, that $\mathbf{A}$ is $\aleph_0$-categorical, can be dropped.