Algorithm for Constraint Satisfaction Problem

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where $\rho_1, \dots, \rho_s \in \Gamma$.
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Task

Find an algorithm that solves CSP(Γ) in polynomial time if Γ is preserved by a WNU operation.

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Over-arching assumption

Every domain D and every relation ρ are viewed as algebras $(D; w)$ and $(\rho; w)$, where w is a special WNU operation of arity m.

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$(a_1,\dots,a_{i-1},a_i,a_{i+1},\dots,a_n) \in \rho$ and $(a_i,b_i) \in \sigma$ implies
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A relation $\rho \subseteq A_1 \times \dots \times A_n$ is subdirect if the $\mathrm{pr}_i(\rho) = A_i$ for every i.

Binary Absorption

B binary absorbs A if there exists a binary term operation $f \in \mathrm{Clo}(w)$ such that $f(B, A) \subseteq B$ and $f(A, B) \subseteq B$.

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Theorem (Absorption implies absorption)

Suppose $\rho \subseteq D \times D_1 \times \cdots \times D_n$ is a relation such that $\mathrm{pr}_1(\rho) = D$, $B(x) = \exists y_1 \ldots \exists y_n\, \rho(x, y_1, \ldots, y_n) \wedge (\forall i : y_i \in B_i)$, where B_i is a binary absorbing subuniverse in D_i with a term operation t for every i. Then B is a binary absorbing subuniverse in D with the term operation t.

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Theorem (Arity of a B-essential relation)

Suppose $\rho \subseteq A^n$, $n \geq 2$, B binary absorbs A, $\rho \cap (B^{i-1} \times A \times B^{n-i}) \neq \varnothing$ for every i. Then $\rho \cap B^n \neq \varnothing$.

$C \subseteq A$ is called a center if there exists an algebra $\mathbf{B} = (B; w_B)$ and a subdirect subalgebra $(R; w_R)$ of $\mathbf{A} \times \mathbf{B}$ such that there is no binary absorption on $\mathbf{B}$ and $C = \{a \in A \mid \forall b \in B \colon (a, b) \in R\}$.

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Theorem (Arity of C-essential relation)

Suppose $\rho \subseteq A^n$, $n \geq 3$, C is a minimal center in A, $\rho \cap (C^{i-1} \times A \times C^{n-i}) \neq \varnothing$ for every i. Then $\rho \cap C^n \neq \varnothing$.

Polynomially Complete Algebras

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Theorem (All relations are binary)

Suppose $\rho \subseteq A_1 \times \cdots \times A_n$ is a subdirect relation, A_i is a PC algebra, there is no binary absorption and center on A_i for every i. Then ρ can be represented as a conjunction of binary rectangular relations.

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Theorem (Compatible with the minimal PC congruence)

Suppose $\rho \subseteq A_1 \times \cdots \times A_n$ is a subdirect relation, the i-th variable of ρ is compatible with the minimal PC congruence on A_i for every $i \geq 2$. Then the first variable of ρ is compatible with the minimal PC congruence on A_1.

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Theorem (Conjunction of linear equations)

Suppose $\rho \subseteq A_1 \times \cdots \times A_n$, A_i is an linear algebra. Then ρ can be represented as a conjunction of linear equations.

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3. there exists a proper congruence σ on A such that $(A; w)/\sigma$ is polynomially complete

We always have something!

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An instance is cycle-consistent if for every variable x_i and $a \in D_i$ any cycle $(x_i =)z_1 - C_1 - z_2 - C_2 - \cdots - z_{n-1} - C_{n-1} - z_n(= x_i)$ connects a and a.

- z_i and z_{i+1} are variables in the scope of the constraint C_i.
- a path $z_1 - C_1 - z_2 - C_2 - \cdots - z_{n-1} - C_{n-1} - z_n$ connects a_1 and a_n if there exist $a_2, \ldots, a_{n-1}$ such that (a_i, a_{i+1}) belongs to the projection of C_i onto (z_i, z_{i+1}).

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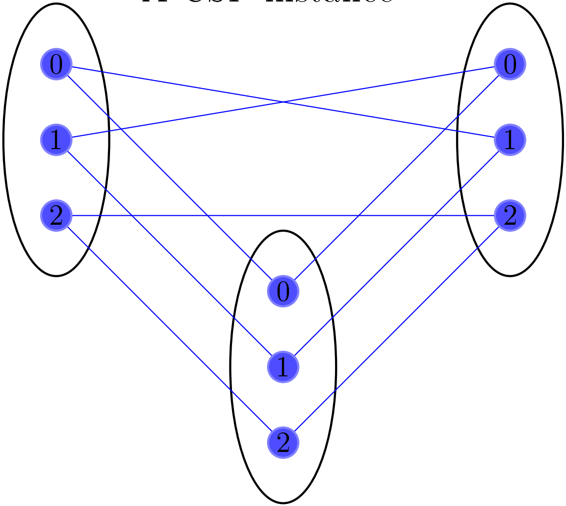
Why cycle-consistency but not (2,3)-consistency?

- Replacement of any constraint by a bigger constraint preserves this consistency!

A CSP instance



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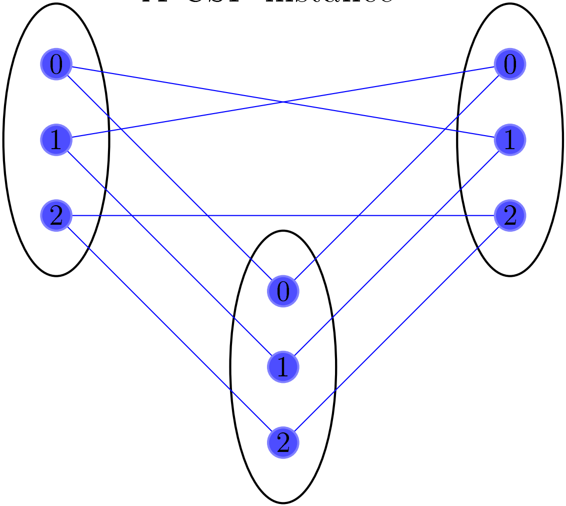

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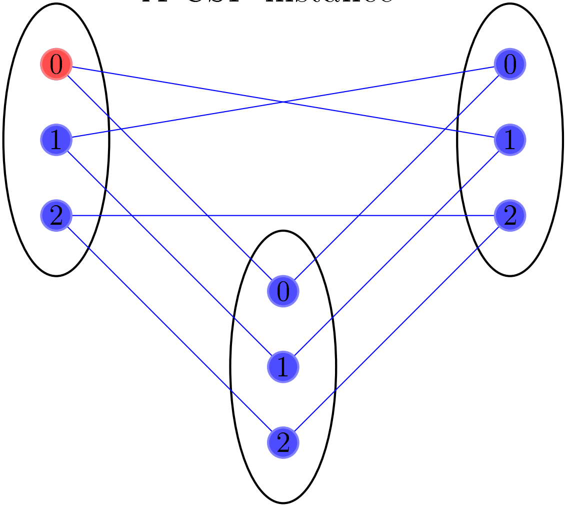

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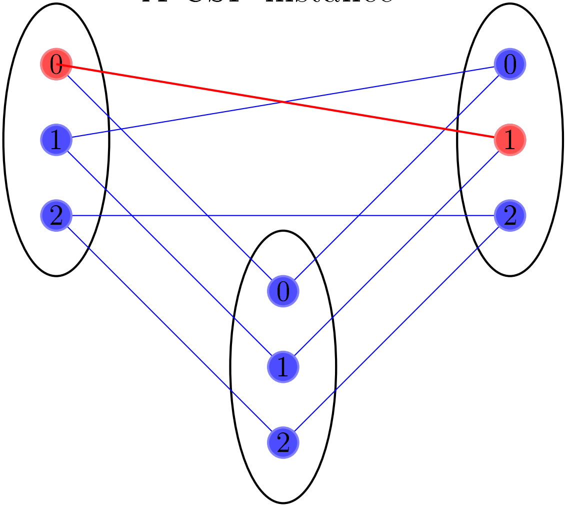

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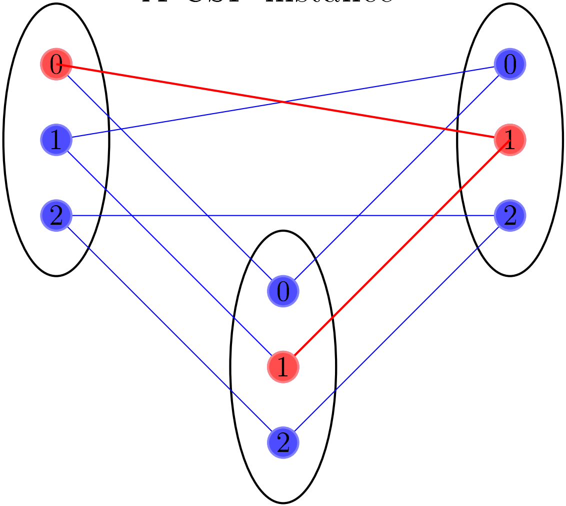

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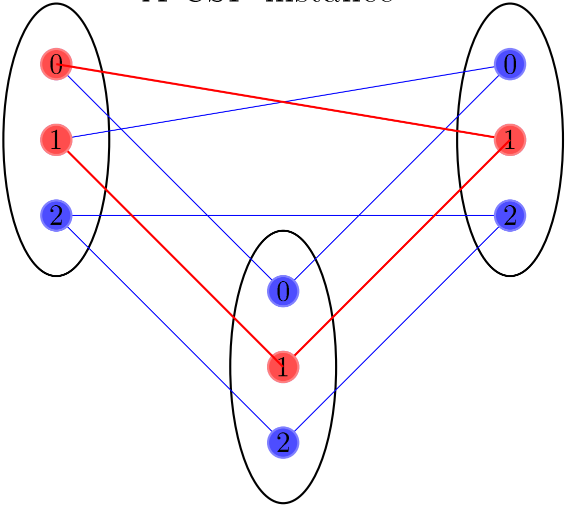

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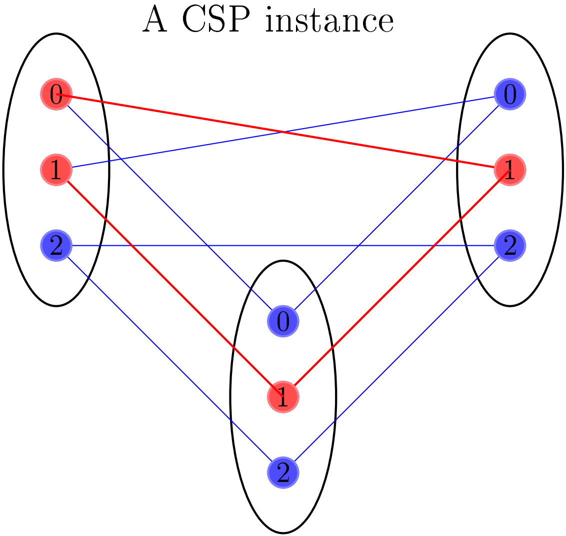

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A CSP instance Θ is fragmented if the set of variables $\mathbf{X}$ can be divided into nonempty sets $\mathbf{X_1}$ and $\mathbf{X_2}$ such that the constraint scope of any constraint either has variables only from $\mathbf{X_1}$, or only from $\mathbf{X_2}$.

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A CSP instance is irreducible if for any subinstance $\mathbf{C}' \subseteq \mathbf{C}$ and any set of variables $\mathbf{X}' \subseteq \mathbf{X}$ the projection of $\mathbf{C}'$ onto $\mathbf{X}'$ is fragmented, linked, or its solution set is subdirect.

We cannot loose the only solution

Theorem 1

Suppose Θ is a cycle-consistent irreducible CSP instance, B is a binary absorbing set of D_i. Then Θ has a solution if and only if Θ has a solution with $x_i \in B$.

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Theorem 3

Suppose Θ is a cycle-consistent irreducible CSP instance,

- no binary absorption and center on D_j for every j,
- $(D_i; w)/\sigma$ is a polynomially complete algebra,
- E is an equivalence class of σ.

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- It returns "no solutions";
- It returns "there exists a solution".

System of Equations in $\mathbb{Z}_4$.

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 0 \\ 2x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 = 2 \\ x_1 + x_2 + 2x_4 = 2 \end{cases}$$

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- for every (x_i, x_j) calculate the projection $\rho_{i,j}$ of all constraints onto these variables.
- for every i, j, k replace $\rho_{i,j}$ by $\rho'_{i,j}$ where $\rho'_{i,j}(x, y) = \exists z\ \rho_{i,j}(x, y) \wedge \rho_{i,k}(x, z) \wedge \rho_{k,j}(z, y)$.

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- repeat this procedure while possible
- if $\mathrm{pr}_1(\rho_{i,j}) \neq D_i$ or $\mathrm{pr}_2(\rho_{i,j}) \neq D_j$, then reduce the domain.

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  2. Choose a constraint C having the variable x_i in the scope for some $i \in I$, choose another variable x_j from the scope such that $j \notin I$.
  3. Denote the projection of C onto (x_i, x_j) by δ.

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  2. Choose a constraint C having the variable x_i in the scope for some $i \in I$, choose another variable x_j from the scope such that $j \notin I$.
  3. Denote the projection of C onto (x_i, x_j) by δ.
  4. Put $\sigma_j(x, y) = \exists x' \exists y' \delta(x, x') \wedge \delta(y, y') \wedge \sigma_i(x', y')$. If σ_j is a proper congruence, then add j to I.

Step 2

Check whether Θ is irreducible. If not, then we reduce a domain D_i for some i or state that there are no solutions.

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  5. go to the next C, x_i, and x_j in 2).

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  5. go to the next C, x_i, and x_j in 2).
- Consider the projection of the instance onto $\mathbf{X}' := \{x_i \mid i \in I\}$.
- For every equivalence class of σ_k we have an instance on a smaller domain.
- Solve every instance on a smaller domain and check whether the solution set of the projection onto $\mathbf{X}'$ is subdirect.

Step 3

Weaken Every Constraint

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- Check that the obtained instance has a solution with $x_i = b$ for every $i \in \{1, 2, \ldots, n\}$ and $b \in D_i$.

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- I believe this step can be omitted.

Reduction of a domain

Step 4

If D_i has a binary absorbing subuniverse $B_i \subsetneq D_i$ for some i, then we reduce D_i to B_i.

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If there exists a congruence σ on D_i such that the algebra $(D_i; w)/\sigma$ is polynomially complete, then we reduce D_i to any equivalence class of σ.

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If there exists a congruence σ on D_i such that the algebra $(D_i; w)/\sigma$ is polynomially complete, then we reduce D_i to any equivalence class of σ.

- We cannot loose the only solution.

Linear Case

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- If $|D_i| > 1$ then σ_i is proper.

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- Define Θ_L with domains $L_1, \ldots, L_n$, variables $x'_1, \ldots, x'_n$, and constraints factorized by $\sigma_1, \ldots, \sigma_n$.

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- Define Θ_L with domains $L_1, \ldots, L_n$, variables $x'_1, \ldots, x'_n$, and constraints factorized by $\sigma_1, \ldots, \sigma_n$.
- Every relation on $\mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_l}$ preserved by $z_1 + \ldots + z_m$ is known to be a conjunction of linear equations.
- The instance Θ_L can be viewed as a system of linear equations in $\mathbb{Z}_p$ for different p.

System of Equations in $\mathbb{Z}_4$.

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 0 \\ 2x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 = 2 \\ x_1 + x_2 + 2x_4 = 2 \end{cases}$$

System of Equations in $\mathbb{Z}_2$.

$$\begin{cases} x_1' + x_3' + x_4' = 0 \\ x_2' + x_3' + x_4' = 0 \\ x_1' + x_2' = 0 \\ x_1' + x_2' = 0 \end{cases}$$

$x_i' = x_i \mod 2$.

Let Eq be a set of constraints on $L_1, \ldots, L_n$ (which are linear equations).

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- Solve system of linear equations $\Theta_L \cup Eq$ and choose independent variables $y_1, \dots, y_k$.

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- Solve system of linear equations $\Theta_L \cup Eq$ and choose independent variables $y_1, \ldots, y_k$.
- If it has no solutions then Θ has no solutions.
- If it has just one solution, then solve the reduction of Θ to this solution. Either we get a solution of Θ, or Θ has no solutions.

System of Equations in $\mathbb{Z}_4$.

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 0 \\ 2x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 = 2 \\ x_1 + x_2 + 2x_4 = 2 \end{cases}$$

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Independent variables: x_1', x_3'.

General solution:

$x_1' = x_1', x_2' = x_1', x_3' = x_3'$,

$x_4' = x_1' + x_3'$.

- There exists a linear mapping $\varphi\colon \mathbb{Z}_{q_1} \times \cdots \times \mathbb{Z}_{q_k} \to L_1 \times \cdots \times L_n$ such that any solution of $\Theta_L \cup Eq$ can be obtained as $\varphi(a_1, \ldots, a_k)$ for some $(a_1, \ldots, a_k)$.

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- We can check whether we have a solution in $\varphi(a_1, \ldots, a_k)$: just solve an instance on a smaller domain.
- We can check whether we have a solution in $\varphi(a_1, \ldots, a_k)$ for every $(a_1, \ldots, a_k)$:
  we just need to check the existence of a solution in $\varphi(0, \ldots, 0)$ and $\varphi(0, \ldots, 0, 1, 0, \ldots, 0)$ for any position of 1.

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Step 9

If Θ has a solution in $\varphi(0, \ldots, 0)$, then Θ has a solution.

System of Equations in $\mathbb{Z}_4$.

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 0 \\ 2x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 = 2 \\ x_1 + x_2 + 2x_4 = 2 \end{cases}$$

$x_i' = x_i \mod 2$.

System of Equations in $\mathbb{Z}_2$.

$$\begin{cases} x_1' + x_3' + x_4' = 0 \\ x_2' + x_3' + x_4' = 0 \\ x_1' + x_2' = 0 \\ x_1' + x_2' = 0 \end{cases}$$

Independent variables: x_1', x_3'.

General solution:

$x_1' = x_1', x_2' = x_1', x_3' = x_3'$,
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$(x_1', x_3') = (0, 0) \Rightarrow (0, 0, 0, 0)$

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No corresponding solutions.

$x'_i = x_i \mod 2$.

System of Equations in $\mathbb{Z}_2$.

$$\begin{cases} x'_1 + x'_3 + x'_4 = 0 \\ x'_2 + x'_3 + x'_4 = 0 \\ x'_1 + x'_2 = 0 \\ x'_1 + x'_2 = 0 \end{cases}$$

Independent variables: x'_1, x'_3.

General solution:

$x'_1 = x'_1, x'_2 = x'_1, x'_3 = x'_3$,

$x'_4 = x'_1 + x'_3$.

$(x'_1, x'_3) = (0, 0) \Rightarrow (0, 0, 0, 0)$

Step 10

Put $\Theta' := \Theta$. Remove from Θ' all constraints that are weaker than some other constraints of Θ'.

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Step 11

For every constraint $C \in \Theta'$

1. Let Ω be obtained from Θ' by replacing of the constraint $C \in \Theta'$ by all weaker constraints without dummy variables. Remove from Ω all constraints that are weaker than some other constraints of Ω.

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2. If Ω has no solutions in $\varphi(b_1, \ldots, b_k)$ for some $(b_1, \ldots, b_k)$, then put $\Theta' := \Omega$. Repeat Step 11.

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2. If Ω has no solutions in $\varphi(b_1, \ldots, b_k)$ for some $(b_1, \ldots, b_k)$, then put $\Theta' := \Omega$. Repeat Step 11.

- Θ' has no solutions in $\varphi(b_1, \ldots, b_k)$ for some $(b_1, \ldots, b_k)$;
- if we replace any constraint $C \in \Theta'$ by all weaker constraints, then we get an instance that has a solution in $\varphi(a_1, \ldots, a_k)$ for every $(a_1, \ldots, a_k)$.

System of Equations in $\mathbb{Z}_4$.

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 0 \\ 2x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 = 2 \\ x_1 + x_2 + 2x_4 = 2 \end{cases}$$

No corresponding solutions.

$x_i' = x_i \mod 2$.

System of Equations in $\mathbb{Z}_2$.

$$\begin{cases} x_1' + x_3' + x_4' = 0 \\ x_2' + x_3' + x_4' = 0 \\ x_1' + x_2' = 0 \\ x_1' + x_2' = 0 \end{cases}$$

Independent variables: x_1', x_3'.

General solution:

$x_1' = x_1', x_2' = x_1', x_3' = x_3'$,

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$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 0 \\ 2x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 = 2 \end{cases}$$

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Step 12

Suppose Θ' is not linked.

Step 12

Suppose Θ' is not linked. For each i from 1 to k

1. Check that for every $(a_1, \ldots, a_i)$ there exist $(a_{i+1}, \ldots, a_k)$ and a solution of Θ' in $\varphi(a_1, \ldots, a_k)$.

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2. If yes, go to the next i.

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2. If yes, go to the next i.
3. If no, then find an equation $c_1 y_1 + \cdots + c_i y_i = c_0$ such that for every $(a_1, \ldots, a_i)$ satisfying $c_1 a_1 + \cdots + c_i a_i = c_0$ there exist $(a_{i+1}, \ldots, a_k)$ and a solution of Θ' in $\varphi(a_1, \ldots, a_k)$.

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1. Check that for every $(a_1, \ldots, a_i)$ there exist $(a_{i+1}, \ldots, a_k)$ and a solution of Θ' in $\varphi(a_1, \ldots, a_k)$.
2. If yes, go to the next i.
3. If no, then find an equation $c_1 y_1 + \cdots + c_i y_i = c_0$ such that for every $(a_1, \ldots, a_i)$ satisfying $c_1 a_1 + \cdots + c_i a_i = c_0$ there exist $(a_{i+1}, \ldots, a_k)$ and a solution of Θ' in $\varphi(a_1, \ldots, a_k)$.
4. Add the equation $c_1 y_1 + \cdots + c_i y_i = c_0$ to Eq.
5. Go to Step 8.

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System of Equations in $\mathbb{Z}_4$.

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 0 \\ 2x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 = 2 \\ \end{cases}$$

No corresponding solutions.

$x_i' = x_i \mod 2$.

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Independent variables: x_1', x_3'.

General solution:

$x_1' = x_1', x_2' = x_1', x_3' = x_3'$,
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$(x_1', x_3') = (0, 0) \Rightarrow (0, 0, 0, 0)$

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General solution:

$x_1' = x_1', x_2' = x_1', x_3' = x_3'$,

$x_4' = x_1' + x_3'$.

$(x_1', x_3') = (0, 0) \Rightarrow (0, 0, 0, 0)$

Find a new equation

$c_1 x_1' + c_3 x_3' = c_0$

System of Equations in $\mathbb{Z}_4$.

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 0 \\ 2x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 = 2 \\ \end{cases}$$

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Independent variables: x_1', x_3'.

General solution:

$x_1' = x_1', x_2' = x_1', x_3' = x_3'$,
$x_4' = x_1' + x_3'$.
$(x_1', x_3') = (0,0) \Rightarrow (0,0,0,0)$
Find a new equation
$c_1 x_1' + c_3 x_3' = c_0$
$(x_1', x_3') = (1,0)$

System of Equations in $\mathbb{Z}_4$.

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 0 \\ 2x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 = 2 \end{cases}$$

No corresponding solutions.

$x_i' = x_i \mod 2$.

System of Equations in $\mathbb{Z}_2$.

$$\begin{cases} x_1' + x_3' + x_4' = 0 \\ x_2' + x_3' + x_4' = 0 \\ x_1' + x_2' = 0 \\ x_1' + x_2' = 0 \end{cases}$$

Independent variables: x_1', x_3'.

General solution:

$x_1' = x_1', x_2' = x_1', x_3' = x_3',$

$x_4' = x_1' + x_3'.$

$(x_1', x_3') = (0, 0) \Rightarrow (0, 0, 0, 0)$

Find a new equation

$c_1 x_1' + c_3 x_3' = c_0$

$(x_1', x_3') = (1, 0) \Rightarrow (1, 1, 0, 1)$

System of Equations in $\mathbb{Z}_4$.

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 0 \\ 2x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 = 2 \end{cases}$$

No corresponding solutions.

$(x_1, x_2, x_3, x_4) = (1, 1, 0, 1)$

$x_i' = x_i \mod 2$.

System of Equations in $\mathbb{Z}_2$.

$$\begin{cases} x_1' + x_3' + x_4' = 0 \\ x_2' + x_3' + x_4' = 0 \\ x_1' + x_2' = 0 \\ x_1' + x_2' = 0 \end{cases}$$

Independent variables: x_1', x_3'.

General solution:

$x_1' = x_1', x_2' = x_1', x_3' = x_3',$

$x_4' = x_1' + x_3'.$

$(x_1', x_3') = (0, 0) \Rightarrow (0, 0, 0, 0)$

Find a new equation

$c_1 x_1' + c_3 x_3' = c_0$

$(x_1', x_3') = (1, 0) \Rightarrow (1, 1, 0, 1)$

System of Equations in $\mathbb{Z}_4$.

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 0 \\ 2x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 = 2 \end{cases}$$

No corresponding solutions.

$(x_1, x_2, x_3, x_4) = (1, 1, 0, 1)$

$x_i' = x_i \mod 2.$

System of Equations in $\mathbb{Z}_2$.

$$\begin{cases} x_1' + x_3' + x_4' = 0 \\ x_2' + x_3' + x_4' = 0 \\ x_1' + x_2' = 0 \\ x_1' + x_2' = 0 \end{cases}$$

Independent variables: x_1', x_3'.

General solution:

$x_1' = x_1', x_2' = x_1', x_3' = x_3',$
$x_4' = x_1' + x_3'.$

$(x_1', x_3') = (0, 0) \Rightarrow (0, 0, 0, 0)$

Find a new equation

$c_1 x_1' + c_3 x_3' = c_0$

$(x_1', x_3') = (1, 0) \Rightarrow (1, 1, 0, 1)$

$(x_1', x_3') = (0, 1)$

System of Equations in $\mathbb{Z}_4$.

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 0 \\ 2x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 = 2 \end{cases}$$

No corresponding solutions.

$(x_1, x_2, x_3, x_4) = (1, 1, 0, 1)$

$x'_i = x_i \mod 2$.

System of Equations in $\mathbb{Z}_2$.

$$\begin{cases} x'_1 + x'_3 + x'_4 = 0 \\ x'_2 + x'_3 + x'_4 = 0 \\ x'_1 + x'_2 = 0 \\ x'_1 + x'_2 = 0 \end{cases}$$

Independent variables: x'_1, x'_3.

General solution:

$x'_1 = x'_1, x'_2 = x'_1, x'_3 = x'_3,$

$x'_4 = x'_1 + x'_3$.

$(x'_1, x'_3) = (0, 0) \Rightarrow (0, 0, 0, 0)$

Find a new equation

$c_1 x'_1 + c_3 x'_3 = c_0$

$(x'_1, x'_3) = (1, 0) \Rightarrow (1, 1, 0, 1)$

$(x'_1, x'_3) = (0, 1) \Rightarrow (0, 0, 1, 1)$

System of Equations in $\mathbb{Z}_4$.

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 0 \\ 2x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 = 2 \\ \end{cases}$$

No corresponding solutions.

$(x_1, x_2, x_3, x_4) = (1, 1, 0, 1)$
No corresponding solutions.
$x_i' = x_i \mod 2$.

System of Equations in $\mathbb{Z}_2$.

$$\begin{cases} x_1' + x_3' + x_4' = 0 \\ x_2' + x_3' + x_4' = 0 \\ x_1' + x_2' = 0 \\ x_1' + x_2' = 0 \end{cases}$$

Independent variables: x_1', x_3'.
General solution:
$x_1' = x_1', x_2' = x_1', x_3' = x_3'$,
$x_4' = x_1' + x_3'$.
$(x_1', x_3') = (0, 0) \Rightarrow (0, 0, 0, 0)$
Find a new equation
$c_1 x_1' + c_3 x_3' = c_0$
$(x_1', x_3') = (1, 0) \Rightarrow (1, 1, 0, 1)$
$(x_1', x_3') = (0, 1) \Rightarrow (0, 0, 1, 1)$

System of Equations in $\mathbb{Z}_4$.

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 0 \\ 2x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 = 2 \end{cases}$$

No corresponding solutions.

$(x_1, x_2, x_3, x_4) = (1, 1, 0, 1)$

No corresponding solutions.

$x_i' = x_i \mod 2$.

System of Equations in $\mathbb{Z}_2$.

$$\begin{cases} x_1' + x_3' + x_4' = 0 \\ x_2' + x_3' + x_4' = 0 \\ x_1' + x_2' = 0 \\ x_1' + x_2' = 0 \end{cases}$$

Independent variables: x_1', x_3'.

General solution:

$x_1' = x_1', x_2' = x_1', x_3' = x_3'$,

$x_4' = x_1' + x_3'$.

$(x_1', x_3') = (0, 0) \Rightarrow (0, 0, 0, 0)$

Find a new equation

$c_1 x_1' + c_3 x_3' = c_0$

$(x_1', x_3') = (1, 0) \Rightarrow (1, 1, 0, 1)$

$(x_1', x_3') = (0, 1) \Rightarrow (0, 0, 1, 1)$

The new equation $x_1' = 1$.

Step 13

Suppose Θ' is linked.

1. Find an equation $c_1 y_1 + \cdots + c_k y_k = c_0$ such that for every $(a_1, \ldots, a_k)$ satisfying $c_1 a_1 + \cdots + c_k a_k = c_0$ there exists a solution of Θ' in $\varphi(a_1, \ldots, a_k)$.

Step 13

Suppose Θ' is linked.

1. Find an equation $c_1 y_1 + \cdots + c_k y_k = c_0$ such that for every $(a_1, \ldots, a_k)$ satisfying $c_1 a_1 + \cdots + c_k a_k = c_0$ there exists a solution of Θ' in $\varphi(a_1, \ldots, a_k)$.
2. If the equation was not found then Θ has no solutions.

Step 13

Suppose Θ' is linked.

1. Find an equation $c_1 y_1 + \cdots + c_k y_k = c_0$ such that for every $(a_1, \ldots, a_k)$ satisfying $c_1 a_1 + \cdots + c_k a_k = c_0$ there exists a solution of Θ' in $\varphi(a_1, \ldots, a_k)$.
2. If the equation was not found then Θ has no solutions.
3. Add the equation $c_1 a_1 + \cdots + c_k a_k = c_0$ to Eq.
4. Go to Step 8.

System of Equations in $\mathbb{Z}_4$.

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 0 \\ 2x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 = 2 \\ x_1 + x_2 + 2x_4 = 2 \end{cases}$$

System of Equations in $\mathbb{Z}_2$.

$$\begin{cases} x_1' + x_3' + x_4' = 0 \\ x_2' + x_3' + x_4' = 0 \\ x_1' + x_2' = 0 \\ x_1' + x_2' = 0 \\ x_1' = 1 \end{cases}$$

$x_i' = x_i \mod 2.$

Step 8

- Solve system of linear equations $\Theta_L \cup Eq$ and choose independent variables $y_1, \ldots, y_k$.
- If it has no solutions then Θ has no solutions.
- If it has just one solution, then solve the reduction of Θ to this solution. Either we get a solution of Θ, or Θ has no solutions.

System of Equations in $\mathbb{Z}_4$.

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 0 \\ 2x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 = 2 \\ x_1 + x_2 + 2x_4 = 2 \end{cases}$$

System of Equations in $\mathbb{Z}_2$.

$$\begin{cases} x_1' + x_3' + x_4' = 0 \\ x_2' + x_3' + x_4' = 0 \\ x_1' + x_2' = 0 \\ x_1' + x_2' = 0 \\ x_1' = 1 \end{cases}$$

$x_i' = x_i \mod 2$.

System of Equations in $\mathbb{Z}_4$.

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 0 \\ 2x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 = 2 \\ x_1 + x_2 + 2x_4 = 2 \end{cases}$$

$x_i' = x_i \mod 2$.

System of Equations in $\mathbb{Z}_2$.

$$\begin{cases} x_1' + x_3' + x_4' = 0 \\ x_2' + x_3' + x_4' = 0 \\ x_1' + x_2' = 0 \\ x_1' + x_2' = 0 \\ x_1' = 1 \end{cases}$$

Independent variable: x_3'.

General solution:

$x_1' = 1, x_2' = 1, x_3' = x_3'$,
$x_4' = 1 + x_3'$.

- There exists a linear mapping $\varphi\colon \mathbb{Z}_{q_1} \times \cdots \times \mathbb{Z}_{q_k} \to L_1 \times \cdots \times L_n$ such that any solution of $\Theta_L \cup Eq$ can be obtained as $\varphi(a_1, \ldots, a_k)$ for some $(a_1, \ldots, a_k)$.
- We can check whether we have a solution in $\varphi(a_1, \ldots, a_k)$: just solve an instance on a smaller domain.
- We can check whether we have a solution in $\varphi(a_1, \ldots, a_k)$ for every $(a_1, \ldots, a_k)$: we just need to check the existence of a solution in $\varphi(0, \ldots, 0)$ and $\varphi(0, \ldots, 0, 1, 0, \ldots, 0)$ for any position of 1.

Step 9

If Θ has a solution in $\varphi(0, \ldots, 0)$, then Θ has a solution.

System of Equations in $\mathbb{Z}_4$.

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 0 \\ 2x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 = 2 \\ x_1 + x_2 + 2x_4 = 2 \end{cases}$$

$x_i' = x_i \mod 2$.

System of Equations in $\mathbb{Z}_2$.

$$\begin{cases} x_1' + x_3' + x_4' = 0 \\ x_2' + x_3' + x_4' = 0 \\ x_1' + x_2' = 0 \\ x_1' + x_2' = 0 \\ x_1' = 1 \end{cases}$$

Independent variable: x_3'.

General solution:

$x_1' = 1, x_2' = 1, x_3' = x_3'$,
$x_4' = 1 + x_3'$.

System of Equations in $\mathbb{Z}_4$.

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 0 \\ 2x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 = 2 \\ x_1 + x_2 + 2x_4 = 2 \end{cases}$$

$x_i' = x_i \mod 2$.

System of Equations in $\mathbb{Z}_2$.

$$\begin{cases} x_1' + x_3' + x_4' = 0 \\ x_2' + x_3' + x_4' = 0 \\ x_1' + x_2' = 0 \\ x_1' + x_2' = 0 \\ x_1' = 1 \end{cases}$$

Independent variable: x_3'.

General solution:

$x_1' = 1, x_2' = 1, x_3' = x_3'$,

$x_4' = 1 + x_3'$.

$x_3' = 0$

System of Equations in $\mathbb{Z}_4$.

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 0 \\ 2x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 = 2 \\ x_1 + x_2 + 2x_4 = 2 \end{cases}$$

$x_i' = x_i \mod 2$.

System of Equations in $\mathbb{Z}_2$.

$$\begin{cases} x_1' + x_3' + x_4' = 0 \\ x_2' + x_3' + x_4' = 0 \\ x_1' + x_2' = 0 \\ x_1' + x_2' = 0 \\ x_1' = 1 \end{cases}$$

Independent variable: x_3'.

General solution:

$x_1' = 1, x_2' = 1, x_3' = x_3'$,

$x_4' = 1 + x_3'$.

$x_3' = 0 \Rightarrow (1, 1, 0, 1)$

System of Equations in $\mathbb{Z}_4$.

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 0 \\ 2x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 = 2 \\ x_1 + x_2 + 2x_4 = 2 \end{cases}$$

No corresponding solutions.

$x_i' = x_i \mod 2$.

System of Equations in $\mathbb{Z}_2$.

$$\begin{cases} x_1' + x_3' + x_4' = 0 \\ x_2' + x_3' + x_4' = 0 \\ x_1' + x_2' = 0 \\ x_1' + x_2' = 0 \\ x_1' = 1 \end{cases}$$

Independent variable: x_3'.

General solution:

$x_1' = 1, x_2' = 1, x_3' = x_3',$

$x_4' = 1 + x_3'.$

$x_3' = 0 \Rightarrow (1, 1, 0, 1)$

Step 10

Put $\Theta' := \Theta$. Remove from Θ' all constraints that are weaker than some other constraints of Θ'.

Step 11

For every constraint $C \in \Theta'$

1. Let Ω be obtained from Θ' by replacing of a constraint $C \in \Theta'$ by all weaker constraints without dummy variables. Remove from Ω all constraints that are weaker than some other constraints of Ω.
2. If Ω has no solutions in $\varphi(a_1, \dots, a_k)$ for some $(a_1, \dots, a_k)$, then put $\Theta' := \Omega$. Repeat Step 11.

- Θ' has no solutions in $\varphi(b_1, \dots, b_k)$ for some $(b_1, \dots, b_k)$;
- if we replace any constraint $C \in \Theta'$ by all weaker constraints, then we get an instance that has a solution in $\varphi(a_1, \dots, a_k)$ for every $(a_1, \dots, a_k)$.

System of Equations in $\mathbb{Z}_4$.

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 0 \\ 2x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 = 2 \\ x_1 + x_2 + 2x_4 = 2 \end{cases}$$

No corresponding solutions.

$x_i' = x_i \mod 2$.

System of Equations in $\mathbb{Z}_2$.

$$\begin{cases} x_1' + x_3' + x_4' = 0 \\ x_2' + x_3' + x_4' = 0 \\ x_1' + x_2' = 0 \\ x_1' + x_2' = 0 \\ x_1' = 1 \end{cases}$$

Independent variable: x_3'.

General solution:

$x_1' = 1, x_2' = 1, x_3' = x_3'$,

$x_4' = 1 + x_3'$.

$x_3' = 0 \Rightarrow (1, 1, 0, 1)$

System of Equations in $\mathbb{Z}_4$.

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 0 \\ 2x_1 + x_2 + x_3 + x_4 = 0 \\ \\ x_1 + x_2 + 2x_4 = 2 \end{cases}$$

No corresponding solutions.

$x_i' = x_i \mod 2$.

System of Equations in $\mathbb{Z}_2$.

$$\begin{cases} x_1' + x_3' + x_4' = 0 \\ x_2' + x_3' + x_4' = 0 \\ x_1' + x_2' = 0 \\ x_1' + x_2' = 0 \\ x_1' = 1 \end{cases}$$

Independent variable: x_3'.

General solution:

$x_1' = 1, x_2' = 1, x_3' = x_3'$,

$x_4' = 1 + x_3'$.

$x_3' = 0 \Rightarrow (1, 1, 0, 1)$

Step 12

Suppose Θ' is not linked. For each i from 1 to k

1. Check that for every $(a_1, \ldots, a_i)$ there exist $(a_{i+1}, \ldots, a_k)$ and a solution of Θ' in $\varphi(a_1, \ldots, a_k)$.
2. If yes, go to the next i.
3. If no, then find an equation $c_1 y_1 + \cdots + c_i y_i = c_0$ such that for every $(a_1, \ldots, a_i)$ satisfying $c_1 a_1 + \cdots + c_i a_i = c_0$ there exist $(a_{i+1}, \ldots, a_k)$ and a solution of Θ' in $\varphi(a_1, \ldots, a_k)$.
4. Add the equation $c_1 y_1 + \cdots + c_i y_i = c_0$ to Eq.
5. Go to Step 8.

Step 13

Suppose Θ' is linked.

Step 13

Suppose Θ' is linked.

1. Find an equation $c_1 y_1 + \cdots + c_k y_k = c_0$ such that for every $(a_1, \ldots, a_k)$ satisfying $c_1 a_1 + \cdots + c_k a_k = c_0$ there exists a solution of Θ' in $\varphi(a_1, \ldots, a_k)$.

System of Equations in $\mathbb{Z}_4$.

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 0 \\ 2x_1 + x_2 + x_3 + x_4 = 0 \\ \\ x_1 + x_2 + 2x_4 = 2 \end{cases}$$

No corresponding solutions.

$x_i' = x_i \mod 2$.

System of Equations in $\mathbb{Z}_2$.

$$\begin{cases} x_1' + x_3' + x_4' = 0 \\ x_2' + x_3' + x_4' = 0 \\ x_1' + x_2' = 0 \\ x_1' + x_2' = 0 \\ x_1' = 1 \end{cases}$$

Independent variable: x_3'.

General solution:

$x_1' = 1, x_2' = 1, x_3' = x_3',$
$x_4' = 1 + x_3'$.
$x_3' = 0 \Rightarrow (1, 1, 0, 1)$

System of Equations in $\mathbb{Z}_4$.

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 0 \\ 2x_1 + x_2 + x_3 + x_4 = 0 \\ \\ x_1 + x_2 + 2x_4 = 2 \end{cases}$$

No corresponding solutions.

$x_i' = x_i \mod 2$.

System of Equations in $\mathbb{Z}_2$.

$$\begin{cases} x_1' + x_3' + x_4' = 0 \\ x_2' + x_3' + x_4' = 0 \\ x_1' + x_2' = 0 \\ x_1' + x_2' = 0 \\ x_1' = 1 \end{cases}$$

Independent variable: x_3'.

General solution:

$x_1' = 1, x_2' = 1, x_3' = x_3'$,

$x_4' = 1 + x_3'$.

$x_3' = 0 \Rightarrow (1, 1, 0, 1)$

Find a new equation $c_3 x_3' = c_0$

System of Equations in $\mathbb{Z}_4$.

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 0 \\ 2x_1 + x_2 + x_3 + x_4 = 0 \\ \\ x_1 + x_2 + 2x_4 = 2 \end{cases}$$

No corresponding solutions.

$x_i' = x_i \mod 2$.

System of Equations in $\mathbb{Z}_2$.

$$\begin{cases} x_1' + x_3' + x_4' = 0 \\ x_2' + x_3' + x_4' = 0 \\ x_1' + x_2' = 0 \\ x_1' + x_2' = 0 \\ x_1' = 1 \end{cases}$$

Independent variable: x_3'.

General solution:

$x_1' = 1, x_2' = 1, x_3' = x_3'$,

$x_4' = 1 + x_3'$.

$x_3' = 0 \Rightarrow (1, 1, 0, 1)$

Find a new equation $c_3 x_3' = c_0$

$x_3' = 1$

System of Equations in $\mathbb{Z}_4$.

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 0 \\ 2x_1 + x_2 + x_3 + x_4 = 0 \\ \\ x_1 + x_2 + 2x_4 = 2 \end{cases}$$

No corresponding solutions.

$x_i' = x_i \mod 2$.

System of Equations in $\mathbb{Z}_2$.

$$\begin{cases} x_1' + x_3' + x_4' = 0 \\ x_2' + x_3' + x_4' = 0 \\ x_1' + x_2' = 0 \\ x_1' + x_2' = 0 \\ x_1' = 1 \end{cases}$$

Independent variable: x_3'.

General solution:

$x_1' = 1, x_2' = 1, x_3' = x_3'$,

$x_4' = 1 + x_3'$.

$x_3' = 0 \Rightarrow (1, 1, 0, 1)$

Find a new equation $c_3 x_3' = c_0$

$x_3' = 1 \Rightarrow (1, 1, 1, 0)$

System of Equations in $\mathbb{Z}_4$.

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 0 \\ 2x_1 + x_2 + x_3 + x_4 = 0 \\ \\ x_1 + x_2 + 2x_4 = 2 \end{cases}$$

No corresponding solutions.

$(x_1, x_2, x_3, x_4) = (1, 1, 1, 0)$

$x_i' = x_i \mod 2$.

System of Equations in $\mathbb{Z}_2$.

$$\begin{cases} x_1' + x_3' + x_4' = 0 \\ x_2' + x_3' + x_4' = 0 \\ x_1' + x_2' = 0 \\ x_1' + x_2' = 0 \\ x_1' = 1 \end{cases}$$

Independent variable: x_3'.

General solution:

$x_1' = 1, x_2' = 1, x_3' = x_3'$,

$x_4' = 1 + x_3'$.

$x_3' = 0 \Rightarrow (1, 1, 0, 1)$

Find a new equation $c_3 x_3' = c_0$

$x_3' = 1 \Rightarrow (1, 1, 1, 0)$

System of Equations in $\mathbb{Z}_4$.

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 0 \\ 2x_1 + x_2 + x_3 + x_4 = 0 \\ \\ x_1 + x_2 + 2x_4 = 2 \end{cases}$$

No corresponding solutions.

$(x_1, x_2, x_3, x_4) = (1, 1, 1, 0)$

$x_i' = x_i \mod 2$.

System of Equations in $\mathbb{Z}_2$.

$$\begin{cases} x_1' + x_3' + x_4' = 0 \\ x_2' + x_3' + x_4' = 0 \\ x_1' + x_2' = 0 \\ x_1' + x_2' = 0 \\ x_1' = 1 \end{cases}$$

Independent variable: x_3'.

General solution:

$x_1' = 1, x_2' = 1, x_3' = x_3'$,

$x_4' = 1 + x_3'$.

$x_3' = 0 \Rightarrow (1, 1, 0, 1)$

Find a new equation $c_3 x_3' = c_0$

$x_3' = 1 \Rightarrow (1, 1, 1, 0)$

The new equation $x_3' = 1$.

Step 13

Suppose Θ' is linked.

1. Find an equation $c_1 y_1 + \cdots + c_k y_k = c_0$ such that for every $(a_1, \ldots, a_k)$ satisfying $c_1 a_1 + \cdots + c_k a_k = c_0$ there exists a solution of Θ' in $\varphi(a_1, \ldots, a_k)$.

Step 13

Suppose Θ' is linked.

1. Find an equation $c_1 y_1 + \cdots + c_k y_k = c_0$ such that for every $(a_1, \ldots, a_k)$ satisfying $c_1 a_1 + \cdots + c_k a_k = c_0$ there exists a solution of Θ' in $\varphi(a_1, \ldots, a_k)$.
2. If the equation was not found then Θ has no solutions.
3. Add the equation $c_1 a_1 + \cdots + c_k a_k = c_0$ to Eq.
4. Go to Step 8.

System of Equations in $\mathbb{Z}_4$.

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 0 \\ 2x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 = 2 \\ x_1 + x_2 + 2x_4 = 2 \end{cases}$$

System of Equations in $\mathbb{Z}_2$.

$$\begin{cases} x'_1 + x'_3 + x'_4 = 0 \\ x'_2 + x'_3 + x'_4 = 0 \\ x'_1 + x'_2 = 0 \\ x'_1 + x'_2 = 0 \\ x'_1 = 1 \\ x'_3 = 1 \end{cases}$$

$x'_i = x_i \mod 2$.

Step 8

- Solve system of linear equations $\Theta_L \cup Eq$ and choose independent variables $y_1, \ldots, y_k$.

System of Equations in $\mathbb{Z}_4$.

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 0 \\ 2x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 = 2 \\ x_1 + x_2 + 2x_4 = 2 \end{cases}$$

System of Equations in $\mathbb{Z}_2$.

$$\begin{cases} x_1' + x_3' + x_4' = 0 \\ x_2' + x_3' + x_4' = 0 \\ x_1' + x_2' = 0 \\ x_1' + x_2' = 0 \\ x_1' = 1 \\ x_3' = 1 \end{cases}$$

$x_i' = x_i \mod 2$.

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The only solution is
$(x_1', x_2', x_3', x_4') = (1, 1, 1, 0)$.

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- Solve system of linear equations $\Theta_L \cup Eq$ and choose independent variables $y_1, \ldots, y_k$.

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- If it has no solutions then Θ has no solutions.
- If it has just one solution, then solve the reduction of Θ to this solution. Either we get a solution of Θ, or Θ has no solutions.

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Ideas of the Proof

The correctness of the algorithm is based on the following facts.

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Theorem 1

Suppose Θ is a cycle-consistent irreducible CSP instance, B is a binary absorbing set of D_i. Then Θ has a solution if and only if Θ has a solution with $x_i \in B$.

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Suppose Θ is a cycle-consistent irreducible CSP instance, B is a binary absorbing set of D_i. Then Θ has a solution if and only if Θ has a solution with $x_i \in B$.

Claim (See the definition of φ in the algorithm)

Suppose Θ' is a linked cycle-consistent irreducible CSP instance,

- Θ' has no solutions in $\varphi(b_1, \ldots, b_k)$ for some $(b_1, \ldots, b_k)$;
- if we replace any constraint $C \in \Theta'$ by all weaker constraints, then we get an instance that has a solution in $\varphi(a_1, \ldots, a_k)$ for every $(a_1, \ldots, a_k)$.

Then Θ' has no solutions, or there exists an equation $c_1 y_1 + \cdots + c_k y_k = c_0$ such that for every $(a_1, \ldots, a_k)$ satisfying $c_1 a_1 + \cdots + c_k a_k = c_0$ there exists a solution of Θ' in $\varphi(a_1, \ldots, a_k)$.

Bridge between Congruences

σ_1 and σ_2 are congruences on D_1 and D_2, correspondingly.
A relation $\rho \subseteq D_1^2 \times D_2^2$ is a bridge from σ_1 to σ_2 if

1. the first two variables of ρ are compatible with σ_1,
2. the last two variables of ρ are compatible with σ_2,
3. $\mathrm{pr}_{1,2}(\rho) \supsetneq \sigma_1$,
4. $\mathrm{pr}_{3,4}(\rho) \supsetneq \sigma_2$,
5. $(a_1, a_2, a_3, a_4) \in \rho$ implies $(a_1, a_2) \in \sigma_1 \Leftrightarrow (a_3, a_4) \in \sigma_2$.

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Congruences σ_1 and σ_2 on a set D are adjacent there exists a reflexive bridge from σ_1 to σ_2.

Examples

- σ is always adjacent with itself:
  $\rho(x_1, x_2, x_3, x_4) = \sigma(x_1, x_3) \wedge \sigma(x_2, x_4)$,

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- Nonreflexive bridge between the equality congruences on $\mathbb{Z}_4$ can be defined by $\{(x_1, x_2, y_1, y_2) \mid x_1 - x_2 = y_1 - y_2 \wedge 2x_1 = 2x_2 = 2y_1 + 2 = 2y_2 = 2\}$

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- Suppose ρ_1 is a bridge from σ_1 to σ_2, ρ_2 is a bridge from σ_2 to σ_3. Then a bridge from σ_1 to σ_3 can be defined by
  $\exists y_1 \exists y_2 \rho_1(x_1, x_2, y_1, y_2) \wedge \rho_2(y_1, y_2, z_1, z_2)$.

By $\mathsf{Con}(\rho, i)$ we denote the binary relation $\sigma(y, y')$ defined by

$$\exists x_1 \ldots \exists x_{i-1} \exists x_{i+1} \ldots \exists x_n\ \rho(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) \wedge \\ \rho(x_1, \ldots, x_{i-1}, y', x_{i+1}, \ldots, x_n).$$

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Example

Suppose a relation δ has parallelogram property.
Then a bridge from $\text{Con}(\delta, 1)$ to $\text{Con}(\delta, 2)$ can be defined by

$$\rho(x_1, x_1', x_2, x_2') = \exists x_3 \ldots \exists x_n\ \delta(x_1, x_2, x_3, \ldots, x_n) \wedge \delta(x_1', x_2', x_3, \ldots, x_n)$$

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- for a relation ρ with parallelogram property $\mathrm{Con}(\rho, i)$ is always a congruence.

For $C = \rho(x_1, \ldots, x_n)$ by $\mathrm{Con}(C, x_i)$ we denote $\mathrm{Con}(\rho, i)$.

Two constraints C_1 and C_2 are adjacent in a common variable x if $\mathrm{Con}(C_1, x)$ and $\mathrm{Con}(C_2, x)$ are adjacent.

Theorem

Suppose $\rho \subseteq D^4$ is a bridge from σ to σ such that $\rho(x, x, y, y)$ defines a full relation. Then there exists an invariant $\zeta \subseteq D \times D \times \mathbb{Z}_p$ such that $(x_1, x_2, 0) \in \zeta \Leftrightarrow (x_1, x_2) \in \sigma$ and $\mathrm{pr}_{1,2}(\zeta) \supsetneq \sigma$.

Almost a Theorem

Suppose we have a CSP instance Θ

1. Θ is cycle-consistent
2. Θ is irreducible
3. Θ is crucial in $D_1', \ldots, D_n'$, i.e. it has no solutions in $D_1', \ldots, D_n'$ but if we weaken any constraint then we get a solution in $D_1', \ldots, D_n'$.

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Then

1. every constraint relation has parallelogram property, i.e. any permutation of variables in ρ gives a relation ρ' satisfying

$$\alpha_1\beta_2, \beta_1\alpha_2, \beta_1\beta_2 \in \rho' \Rightarrow \alpha_1\alpha_2 \in \rho'.$$

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2. all constraints with common variables are adjacent.
3. if Θ is linked then for every congruence $\sigma = \mathrm{Con}(C, x)$ there exists a bridge ρ from σ to σ such that $\rho(x, x, y, y)$ defines a full relation.

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- If $D'_1, \ldots, D'_n$ is an absorbing reduction and Θ has a solution, then the restriction to $D'_1, \ldots, D'_n$ restricts z to a binary absorbing set. Contradiction!!!

Thus we can prove

Theorem 1

Suppose Θ is a cycle-consistent irreducible CSP instance, B is a binary absorbing set of D_i. Then Θ has a solution if and only if Θ has a solution with $x_i \in B$.

Idea for Linear Case

Suppose we have a CSP instance Θ

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We do

- Consider the solution set of Θ', factorize it by σ, remove all dependent variables. This set can be described by a system of linear equations with $y_1, \ldots, y_k, z$.

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We do

- Consider the solution set of Θ', factorize it by σ, remove all dependent variables. This set can be described by a system of linear equations with $y_1, \ldots, y_k, z$.
- If we have more than one equation, then we have an equation without z. Contradiction!

We know

- there exists a solution of Θ_L such that Θ' has no corresponding solution with $z = 0$.
- for every solution of Θ_L we have the corresponding solution of Θ'.

We do

- Consider the solution set of Θ', factorize it by σ, remove all dependent variables. This set can be described by a system of linear equations with $y_1, \ldots, y_k, z$.
- If we have more than one equation, then we have an equation without z. Contradiction!
- If the equation is $z = c$ then no solutions.

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- If the equation is $z = c$ then no solutions.
- If the equation is $c_1 y_1 + \dots + c_k y_k + cz = c_0$, then add $c_1 y_1 + \dots + c_k y_k = c_0$ to Θ_L.

Thus, we can prove the following

Claim (See the definition of φ in the algorithm)

Suppose Θ' is a linked cycle-consistent irreducible CSP instance,

- Θ' has no solutions in $\varphi(b_1, \ldots, b_k)$ for some $(b_1, \ldots, b_k)$;
- if we replace any constraint $C \in \Theta'$ by all weaker constraints, then we get an instance that has a solution in $\varphi(a_1, \ldots, a_k)$ for every $(a_1, \ldots, a_k)$.

Then Θ' has no solutions, or there exists an equation $c_1 y_1 + \cdots + c_k y_k = c_0$ such that for every $(a_1, \ldots, a_k)$ satisfying $c_1 a_1 + \cdots + c_k a_k = c_0$ there exists a solution of Θ' in $\varphi(a_1, \ldots, a_k)$.

Thank you for your attention