Non-associative $\mathbf{MV}$-algebras

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1. NMV-algebras

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- $([0, 1], (x + y) \wedge 1, 1 - x, 0)$ *is an* MV*-algebra.*

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Example 2

- $([0, 1], (x + y) \wedge 1, 1 - x, 0)$ *is an MV-algebra.*
- *If* $(B, \vee, \wedge, ', 0, 1)$ *is a Boolean algebra then* $(B, \vee, ', 0)$ *is an MV-algebra.*

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\neg(\neg x \oplus y) \oplus y &\approx \neg(\neg y \oplus x) \oplus x, \\
\neg x \oplus (\neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus z) \oplus z) &\approx 1,
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where $1 := \neg 0$. *Put*

$$x \to y := \neg x \oplus y \text{ and } x \odot y := \neg(\neg x \oplus \neg y).$$

Connections between MV-algebras and NMV-algebras

Every MV-algebra is an NMV-algebra, but not vice versa.

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Example 4

$(\{0, a, b, c, d, 1\}, \oplus, \neg, 0)$ *with*

$\oplus$	0	a	b	c	d	1	$\neg$
0	0	a	b	c	d	1	1
a	a	d	c	c	1	1	d
b	b	c	d	1	d	1	c
c	c	c	1	1	1	1	b
d	d	1	d	1	1	1	a
1	1	1	1	1	1	1	0

is an NMV-algebra, but not an MV-algebra since

$$(a \oplus a) \oplus b = d \neq c = a \oplus (a \oplus b).$$

2. Interval NMV-algebras

Interval NMV-algebras

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for all $x, y \in [a, 1]$*. Then* $([a, 1], \oplus_a, \neg_a, a)$ *is an* NMV*-algebra if and only if*

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for all $x, y \in [a, 1]$. *Then* $([a, 1], \oplus_a, \neg_a, a)$ *is an* NMV-*algebra if and only if*

$$x \to (y \to a) = y \to (x \to a) \textit{ and}$$

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$$\begin{aligned} x \to (y \to a) &= y \to (x \to a) \text{ and} \\ x \to ((x \to a) \to y) &= 1 \end{aligned}$$

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$$x \to (y \to a) = y \to (x \to a) \text{ and}$$
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for all $x, y \in [a, 1]$*.*

3. Idempotent elements

Idempotent elements

Definition 6

An element a of an NMV*-algebra* $\mathbf{A} = (A, \oplus, \neg, 0)$ *is called* *idempotent* *if* $a \oplus a = a$. *Let* $\mathrm{Id}(\mathbf{A})$ *denote the set of all idempotent elements of* $\mathbf{A}$.

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$$\begin{aligned} 0, 1 &\in \mathsf{Id}(\mathbf{A}), \\ a \in \mathsf{Id}(\mathbf{A}) &\Leftrightarrow \neg a \in \mathsf{Id}(\mathbf{A}), \\ \mathsf{Id}(\mathbf{A} \times \mathbf{B}) &= \mathsf{Id}(\mathbf{A}) \times \mathsf{Id}(\mathbf{B}). \end{aligned}$$

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Examples of derivations

Lemma 10

If $\mathbf{A} = (A, \oplus, \neg, 0)$ *is an* NMV*-algebras and* $d \in \mathrm{Der}(\mathbf{A})$ *then* $d(x) \leq x$ *for all* $x \in A$*.*

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Lemma 12

If $\mathbf{B} = (A, \vee, \wedge, ', 0, 1)$ *is a finite Boolean algebra then the* MV*-algebra* $\mathbf{A} = (A, \vee, ', 0)$ *has exactly* $|A|$ *derivations, namely* $x \mapsto x \wedge a$*,* $a \in A$*.*

Examples of derivations, continued

Proof. The mentioned mappings are $|A|$ pairwise different derivations on $\mathbf{A}$. Now let $d \in \operatorname{Der} \mathbf{A}$. Since d is a homomorphism with respect to $\vee$, it is determined by its values on the atoms of $\mathbf{B}$. Because of $d(x) \leq x$ for all $x \in A$, there are only two possibilities for the value of d on an atom of $\mathbf{B}$. Hence $\mathbf{A}$ has at most $|A|$ derivations completing the proof of the lemma.

5. Congruences, filters

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NMV-*algebras are arithmetical and regular.*

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Theorem 14

NMV*-algebras are arithmetical and regular.*

Corollary 15

If $\mathbf{A} = (A, \oplus, \neg, 0)$ *is an* NMV*-algebra and* $\Theta \in \operatorname{Con} \mathbf{A}$ *then* Θ *is determined by* $[1]\Theta$*.*

Filters

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A filter of an NMV*-algebra* $\mathbf{A} = (A, \oplus, \neg, 0)$ *is a subset* F *of* A *satisfying*

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1 &\in F, \\
x, x \to y \in F &\Rightarrow y \in F, \\
x \to y, y \to x \in F &\Rightarrow (x \to z) \to (y \to z), (z \to x) \to (z \to y) \in F.
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$$\begin{aligned} 1 &\in F, \\ x, x \to y \in F &\Rightarrow y \in F, \\ x \to y, y \to x \in F &\Rightarrow (x \to z) \to (y \to z), (z \to x) \to (z \to y) \in F. \end{aligned}$$

Let $\mathrm{Fil}(\mathbf{A})$ *denote the set of all filters of* $\mathbf{A}$*.*

Congruences and filters

For any NMV-algebra $\mathbf{A} = (A, \oplus, \neg, 0)$, $\Theta \in \operatorname{Con}\mathbf{A}$ and $F \in \operatorname{Fil}\mathbf{A}$ put

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For any NMV-algebra $\mathbf{A} = (A, \oplus, \neg, 0)$, $\Theta \in \operatorname{Con} \mathbf{A}$ and $F \in \operatorname{Fil} \mathbf{A}$ put

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$$\begin{aligned} \mathrm{F}(\Theta) &:= [1]\Theta, \\ \mathrm{C}(F) &:= \{(x, y) \in A^2 \mid x \to y, y \to x \in F\}. \end{aligned}$$

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Theorem 17

If $\mathbf{A}$ *is an* NMV*-algebra then* F *and* C *are mutually inverse isomorphisms between* $(\operatorname{Con} \mathbf{A}, \subseteq)$ *and* $(\operatorname{Fil} \mathbf{A}, \subseteq)$.

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Thank you for your attention!