The global dimension of the algebra of the monoid of all partial functions on an $n$-set

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Monoid algebras

- $M$ - finite monoid.
- $\mathbb{C}M$ - monoid algebra.

$$\mathbb{C}M = \left\{ \sum \alpha_i m_i \mid \alpha_i \in \mathbb{C} \quad m_i \in M \right\}$$

- $\mathbb{C}M$ is usually not a semisimple algebra.

Question

Given an interesting monoid $M$, try to find properties/invariants of $\mathbb{C}M$. 
For our talk:

- $M = \mathcal{PT}_n$. The monoid of all partial functions on $\{1, \ldots, n\}$.
- Invariant = The global dimension.

Goal

Find the global dimension of $\mathcal{CPT}_n$. 
Homological algebra - reminder

- $A$ - finite dimensional algebra over $\mathbb{C}$.
- $A\text{-Mod}$ - The category of finite dimensional $A$-modules.
- Let $M \in A\text{-Mod}$ then

$$\text{Hom}_A(M, -): A\text{-Mod} \to \text{Ab}$$

is a (covariant) left exact functor.

**Definition**

An $A$-module $P$ is called *projective* if $\text{Hom}_A(P, -)$ is an exact functor.
Homological algebra - reminder

**Definition**

Let $M$ be an $A$-module. A *projective resolution* of $M$ is an exact sequence

$$0 \rightarrow P_n \rightarrow \ldots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

where every $P_i$ is projective.

$n = \text{length of the projective resolution.}$

**Definition**

The projective dimension of $M$ is the minimal length of a projective resolution of $M$.

**Example**

$\text{pd}(M) = 0 \iff M$ is projective.
Homological algebra - reminder

- \( \text{Ext}^n(M, -) \) - the \( n \)-th right derived functor of \( \text{Hom}(M, -) \).

Fact

\[
pd(M) = \min \{ m \mid \text{Ext}^{m+1}(M, N) = 0 \text{ for every } N \in A - \text{Mod} \}\]
Definition

The *global dimension* of an algebra $A$ is

$$\text{gl. dim}(A) = \sup\{\text{pd}(M) \mid M \in A - \text{Mod}\}$$
One word on the Quiver

Definition

Let $A$ be an algebra. The quiver of $A$ is a directed graph $Q$ that gives a lot of information on the representations of $A$ (in particular, also on the global dimension).

- **Vertices** - Simple modules.
- **Edges** - The number of edges between $S_1$ to $S_2$ is $\dim \Ext^1(S_1, S_2)$.

I.S. (2016): Computation of the Quiver of $\mathbb{CP}T_n$. 
Known results

- Steinberg (2016): $\text{gl. dim}(\mathbb{C} T_n) = n - 1$
The monoid $\mathcal{PT}_n$.

- Regular.

- The $\mathcal{J}$ order is linear.

- The maximal subgroups are $S_k$ where $0 \leq k \leq n$.

- The structure matrix ("Rees sandwich matrix") of $J_k$ is left invertible over $\mathbb{CS}_k$. 
Upper bound

Theorem (Nico’s Theorem)

Let $M$ be a regular monoid and let $k$ be the longest chain in the $\mathcal{J}$-order. Then $\text{gl. dim}(\mathbb{C}M) \leq 2k$.

If all the structure matrices are left or right invertible, then $\text{gl. dim}(\mathbb{C}M) \leq k$.

- For $M = \mathbb{C} \mathcal{P} \mathcal{T}_n$ this gives $\text{gl. dim}(\mathbb{C} \mathcal{P} \mathcal{T}_n) \leq n$.
- It is easy to show we can ignore the $\mathcal{J}$ class of the zero function, so actually $\text{gl. dim}(\mathbb{C} \mathcal{P} \mathcal{T}_n) \leq n - 1$. 
Main Theorem

**Theorem**

\[ \text{gl. dim}(\mathcal{PT}_n) = n - 1. \]

- It is enough to find a module \( M \) with \( \text{pd}(M) = n - 1 \).
- It is enough to find modules \( M, N \) with \( \text{Ext}^{n-1}(M, N) \neq 0 \).
Let $M$ be a finite monoid. There is a one-to-one correspondence between simple modules of $M$ and simple modules of its maximal subgroups.

- The maximal subgroups of $\mathcal{PT}_n$ are $S_k$ for $0 \leq k \leq n$
- Its irreducible representations can be parameterized by partitions $\alpha \vdash k$ for $0 \leq k \leq n$, or equivalently, by Young diagrams.

\[
\{ S_\alpha \mid \alpha \vdash k \quad 0 \leq k \leq n \}\]

- As there is one-to-one correspondence between simple modules and indecomposable projective modules. We obtain the indecomposable projective modules are also parameterized by Young diagrams.

\[
\{ P_\alpha \mid \alpha \vdash k \quad 0 \leq k \leq n \}\]
Let $\alpha \vdash k$ and $\beta \vdash r$ be two Young diagrams. How many times $S_\alpha$ appears as a Jordan-Hölder factor of $P_\beta$?

$$
\begin{array}{cccccc}
\beta \vdash n & \beta \vdash n-1 & \ldots & \beta \vdash 0 \\
\hline
\begin{array}{cccccc}
* & * & * & * & * & \cdots & * \\
* & * & * & * & * & \cdots & * \\
* & * & * & * & * & \cdots & * \\
* & * & * & * & * & \cdots & * \\
* & * & * & * & * & \cdots & * \\
* & * & * & * & * & \cdots & * \\
\end{array}
\end{array}
$$
Cartan matrix

Proposition (Putcha 1995)

This matrix is unitriangular.

\[
\begin{array}{c|cccc}
\beta \vdash n & \beta \vdash n-1 & \cdots & \beta \vdash 0 \\
\hline
\alpha \vdash n & 0 & 0 & 0 & 0 & \cdots & 0 \\
\alpha \vdash n-1 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
\alpha \vdash 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\end{array}
\]
Question

Let \( \alpha \vdash k \) and \( \beta \vdash r \) be two Young diagrams. How many times \( S_\alpha \) appears as a Jordan-Hölder factor of \( P_\beta \)?

- Define \( E(r, k) \) to be the set of all onto total functions from \( \{1, \ldots, r\} \) to \( \{1, \ldots, k\} \). This is an \( S_k \times S_r \) module via action \((\pi, \tau) \ast f = \pi f \tau^{-1}\).

Proposition (IS)

The number of times that \( S_\alpha \) appears as a J-H factor in \( P_\beta \) is the number of times that \( S^\alpha \otimes S^\beta \) appears as an irreducible constituent in \( E(r, k) \).
**Cartan matrix**

**Proposition (IS 2016)**

*Explicit description of the block diagonal below the main diagonal.*

\[\begin{array}{cccc}
\beta \vdash n & \beta \vdash n - 1 & \ldots & \beta \vdash 0 \\
\hline
\alpha \vdash n & \begin{pmatrix} 0 & 0 & 0 & 0 & \ldots & 0 \\ \end{pmatrix} & \\
\alpha \vdash n - 1 & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \end{pmatrix} & \\
\vdots & \begin{pmatrix} 0 & 0 & 0 \\ \end{pmatrix} & \\
\alpha \vdash 0 & \begin{pmatrix} \ast & \ast & \ast \\ \ast & \ast \\ \ast & \ast \\ \ast & \ast \\ \end{pmatrix} & \\
\end{array}\]
Proposition (IS 2016)

Let $\alpha \vdash k$ and $\beta \vdash k + 1$. The number of times that $S_\alpha$ appears as a J-H factor in $P_\beta$ is the number of times that $S_\beta$ appears as an irreducible constituent in

$$\text{Ind}_{S_{k-1} \times S_2}^{S_{k+1}} (\text{Res}^{S_k}_{S_{k-1}} S_\alpha \otimes \text{tr} S_2)$$

which is the number of ways to obtain $\beta$ from $\alpha$ by removing one box and adding two but not in the same column.
Proposition (IS)

Explicit description of another block diagonal.

$$\begin{pmatrix}
\beta \vdash n & \beta \vdash n-1 & \ldots & \beta \vdash 0 \\
\alpha \vdash n & \alpha \vdash n-1 & \ldots & \alpha \vdash 0 \\
\vdots & \vdots & \ddots & \vdots \\
\alpha \vdash 0 & \alpha \vdash 0 & \ldots & \alpha \vdash 0
\end{pmatrix}$$
Proposition (IS)

Let $\alpha \vdash k$ and $\beta \vdash k + 2$. The number of times that $S^\alpha$ appears as a J-H factor in $P_\beta$ is the number of times that $S^\beta$ appears as an irreducible constituent in

$$\text{Ind}_{S^{k+2}_{k-1} \times S_3}^S (\text{Res}_{S_{k-1}}^S (S^\alpha) \otimes \text{tr}_3) \oplus \text{Ind}_{S^{k+2}_{k-2} \times D_4}^S \text{Res}_{S_{k-2} \times S_2}^S S^\alpha.$$
The projective module of the “dual standard” partition

Conjecture (Walter Mazorchuk)

Consider the projective indecomposable module $P_\beta$ for the partition $\beta = [2, 1^{n-2}]$. It contains only few J-H components.

Proposition (IS)

For $n \geq 3$, the only J-H components of $P_\beta$ are the simples for $[2, 1^{n-2}]$, $[2, 1^{n-3}]$ and $[1^{n-1}]$. Each one with multiplicity 1.

Example ($n = 4$)

The J-H components of $P([\begin{array}{c} \cdot & \cdot \\ \cdot & \cdot \end{array}])$ are $S([\begin{array}{c} \cdot & \cdot \\ \cdot & \cdot \end{array}])$, $S([\begin{array}{c} \cdot & \cdot \\ \cdot & \cdot \end{array}])$ and $S([\begin{array}{c} \cdot \\ \cdot \\ \cdot \\ \cdot \end{array}]).$
Quiver of $\mathbb{CPT}_4$
Homological arguments-example

- Consider the short exact sequence

\[ 0 \rightarrow \text{Rad } P(n) \rightarrow P(n) \rightarrow S(n) \rightarrow 0 \]

- By the above we know that the J-H components of $\text{Rad } P(n)$ are $S(n)$ and $S(n)$.

- Other known facts:
  - $S(n)$ is a projective module.
  - $\text{Ext}^1(S(n), S(n)) = \text{Ext}^1(S(n), S(n)) = 0$

  This implies that $\text{Rad } P(n) = S(n) \oplus S(n)$. 
Consider the short exact sequence

\[ 0 \to S \oplus S \to P \to S \to 0 \]

By the “long exact sequence” Theorem we have that

\[ \text{Ext}^k(S, S) \cong \text{Ext}^{k-1}(S \oplus S, S) \]

\[ \text{Ext}^{k-1}(S \oplus S, S) = \text{Ext}^{k-1}(S, S) \oplus \text{Ext}^{k-1}(S, S) \]

Hence

\[ \text{Ext}^k(S, S) \cong \text{Ext}^{k-1}(S, S) \]
Consider the short exact sequence

$$0 \rightarrow \text{Rad } P([2, 1^{n-2}]) \rightarrow P([2, 1^{n-2}]) \rightarrow S([2, 1^{n-2}]) \rightarrow 0$$

By the above we know that the J-H components of $\text{Rad } P([2, 1^{n-2}])$ are $S([2, 1^{n-3}])$ and $S([1^{n-1}])$.

Other known facts:
- $S([1^{n-1}])$ is a projective module.
- $\text{Ext}^1(S([2, 1^{n-3}]), S([1^{n-1}])) = \text{Ext}^1(S([1^{n-1}]), S([2, 1^{n-3}])) = 0$

This implies that

$$\text{Rad } P([2, 1^{n-2}]) = S([2, 1^{n-3}]) \oplus S([1^{n-1}])$$
Homological arguments

Consider the short exact sequence

$$0 \to S([2, 1^{n-3}]) \oplus S([1^{n-1}]) \to P([2, 1^{n-2}]) \to S([2, 1^{n-2}]) \to 0$$

By the “long exact sequence” Theorem we have that

$$\text{Ext}^k(S([2, 1^{n-2}]), S([1])) \cong \text{Ext}^{k-1}(S([2, 1^{n-3}]) \oplus S([1^{n-1}]), S([1]))$$

$$\text{Ext}^{k-1}(S([2, 1^{n-3}]) \oplus S([1^{n-1}]), S([1])) = \text{Ext}^{k-1}(S([2, 1^{n-3}]), S([1]))$$

Hence

$$\text{Ext}^k(S([2, 1^{n-2}]), S([1])) \cong \text{Ext}^{k-1}(S([2, 1^{n-3}]), S([1]))$$
This implies that

$$\text{pd}(S([2, 1^{n-2}])) = \text{pd}(S([2, 1^{n-2}])) + 1$$

So by easy induction we obtain

$$\text{gl. dim}(\mathbb{C} \mathcal{PT}_n) = n - 1$$

as required.
Thank you!