The global dimension of the algebra of the monoid of all partial functions on an *n*-set

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Monoid algebras

- M - finite monoid.
- $\mathbb{C}M$ - monoid algebra.

$$\mathbb{C}M = \{\sum \alpha_i m_i \mid \alpha_i \in \mathbb{C} \quad m_i \in M\}$$

- $\mathbb{C}M$ is usually not a semisimple algebra.

Question

Given an interesting monoid M, try to find properties\invariants of $\mathbb{C}M$

Goal

For our talk:

- $M = \mathcal{PT}_n$. The monoid of all partial functions on $\{1, \dots, n\}$.
- Invariant = The global dimension.

Goal

Find the global dimension of $\mathbb{C}\mathcal{PT}_n$.

Homological algebra - reminder

- A - finite dimensional algebra over $\mathbb{C}$.
- $A - \mathbf{Mod}$ - The category of finite dimensional A-modules.
- Let $M \in A - \mathbf{Mod}$ then

$$\mathrm{Hom}_A(M, -) : A - \mathbf{Mod} \to \mathbf{Ab}$$

is a (covariant) left exact functor.

Definition

An A-module P is called *projective* if $\mathrm{Hom}_A(P, -)$ is an exact functor.

Homological algebra - reminder

Definition

Let M be an A-module. A *projective resolution* of M is an exact sequence

$$0 \to P_n \to \ldots \to P_1 \to P_0 \to M \to 0$$

where every P_i is projective.
$n =$ length of the projective resolution.

Definition

The projective dimension of M is the minimal length of a projective resolution of M.

Example

$\mathrm{pd}(M) = 0 \iff M$ is projective.

Homological algebra - reminder

- $\text{Ext}^n(M, -)$ - the n-th right derived functor of $\text{Hom}(M, -)$.

Fact

$\text{pd}(M) = \min\{m \mid \text{Ext}^{m+1}(M, N) = 0 \textit{ for every } N \in A - \textbf{Mod}\}$

Homological algebra - reminder

Definition

The *global dimension* of an algebra A is

$$\mathrm{gl.dim}(A) = \sup\{\mathrm{pd}(M) \mid M \in \mathrm{A} - \mathbf{Mod}\}$$

One word on the Quiver

Definition

Let A be an algebra. The quiver of A is a directed graph Q that gives a lot of information on the representations of A (in particular, also on the global dimension).

- Vertices - Simple modules.
- Edges - The number of edges between S_1 to S_2 is $\dim \operatorname{Ext}^1(S_1, S_2)$.

- I.S. (2016): Computation of the Quiver of $\mathbb{C}\mathcal{PT}_n$.

Known results

- Steinberg (2016): $\mathrm{gl.dim}(\mathbb{C}\mathcal{T}_n) = n - 1$
- Margolis, Saliola, Steinberg (2015): Certain results on the global dimension of left regular bands.

$\mathcal{PT}_n$

- The monoid $\mathcal{PT}_n$.
  - Regular.
  - The $\mathcal{J}$ order is linear.
  - The maximal subgroups are S_k where $0 \leq k \leq n$.
  - The structure matrix ("Rees sandwich matrix") of J_k is left invertible over $\mathbb{C}S_k$.

Upper bound

Theorem (Nico's Theorem)

Let M be a regular monoid and let k be the longest chain in the $\mathcal{J}$-order. Then $\operatorname{gl.dim}(\mathbb{C}M) \leq 2k$.
If all the structure matrices are left or right invertible, then $\operatorname{gl.dim}(\mathbb{C}M) \leq k$.

- For $M = \mathbb{C}\mathcal{PT}_n$ this gives $\operatorname{gl.dim}(\mathbb{C}\mathcal{PT}_n) \leq n$.
- It is easy to show we can ignore the $\mathcal{J}$ class of the zero function, so actually $\operatorname{gl.dim}(\mathbb{C}\mathcal{PT}_n) \leq n - 1$.

Main Theorem

Theorem

$\mathrm{gl.dim}(\mathbb{C}\mathcal{PT}_n) = n - 1.$

- It is enough to find a module M with $\mathrm{pd}(M) = n - 1$.
- It is enough to find modules M, N with $\mathrm{Ext}^{n-1}(M, N) \neq 0$.

Munn-Ponizovsky

Theorem (Munn-Ponizovsky)

Let M be a finite monoid. There is a one-to-one correspondence between simple modules of M and simple modules of its maximal subgroups.

- The maximal subgroups of $\mathcal{PT}_n$ are S_k for $0 \leq k \leq n$
- Its irreducible representations can be parameterized by partitions $\alpha \vdash k$ for $0 \leq k \leq n$, or equivalently, by Young diagrams.

$$\{S_\alpha \mid \alpha \vdash k \quad 0 \leq k \leq n\}$$

- As there is one-to-one correspondence between simple modules and indecomposable projective modules. We obtain the indecomposable projective modules are also parameterized by Young diagrams.

$$\{P_\alpha \mid \alpha \vdash k \quad 0 \leq k \leq n\}$$

Cartan matrix

Question

Let $\alpha \vdash k$ and $\beta \vdash r$ be two Young diagrams. How many times S_α appears as a Jordan-Hölder factor of P_β?

$$
\begin{array}{ll}
 & \begin{array}{cccc} \overbrace{\qquad\qquad}^{\beta \vdash n} & \overbrace{\qquad\qquad}^{\beta \vdash n-1} & \cdots & \overbrace{\quad}^{\beta \vdash 0} \end{array} \\
\begin{array}{l} \alpha \vdash n \left\{ \begin{array}{c} \\ \\ \end{array} \right. \\ \alpha \vdash n-1 \left\{ \begin{array}{c} \\ \\ \end{array} \right. \\ \\ \vdots \qquad \vdots \\ \alpha \vdash 0 \quad \{ \end{array} &
\left(\begin{array}{cc|cc|cc|c|c}
* & * & * & * & * & * & \cdots & * \\
 & & & & & & & \\
* & * & * & * & * & * & & * \\
\hline
* & * & * & * & & & & \\
* & * & * & * & & & & \\
\hline
* & * & & & & & & \\
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\vdots & & & & & & \ddots & \\
\hline
* & * & & & & & & *
\end{array} \right)
\end{array}
$$

Cartan matrix

Proposition (Putcha 1995)

This matrix is unitriangular.

$$
\begin{array}{cc}
 & \overbrace{\qquad\qquad}^{\beta \vdash n} \quad \overbrace{\qquad\qquad}^{\beta \vdash n-1} \quad \cdots \quad \overbrace{\quad}^{\beta \vdash 0} \\
\begin{array}{c}
\alpha \vdash n \; \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. \\
\alpha \vdash n-1 \; \left\{ \begin{array}{c} \\ \\ \end{array} \right. \\
\vdots \qquad \vdots \\
\alpha \vdash 0 \; \{
\end{array}
&
\left(
\begin{array}{cc|cc|cc|c|c}
 & & 0 & 0 & 0 & 0 & \cdots & 0 \\
 & I & & & & & & \\
 & & 0 & 0 & 0 & 0 & & 0 \\
\hline
* & * & \ddots & & 0 & 0 & & 0 \\
* & * & & & 0 & 0 & & 0 \\
\hline
* & * & & & \ddots & & & \\
\hline
\vdots & & & & & & \ddots & \\
\hline
* & * & & & & & & I
\end{array}
\right)
\end{array}
$$

Cartan matrix

Question

Let $\alpha \vdash k$ and $\beta \vdash r$ be two Young diagrams. How many times S_α appears as a Jordan-Hölder factor of P_β?

- Define $E(r,k)$ to be the set of all onto **total** functions from $\{1,\ldots,r\}$ to $\{1,\ldots,k\}$. This is an $S_k \times S_r$ module via action $(\pi,\tau) * f = \pi f \tau^{-1}$.

Proposition (IS)

The number of times that S_α appears as a J-H factor in P_β is the number of times that $S^\alpha \otimes S^\beta$ appears as an irreducible constituent in $E(r,k)$

Cartan matrix

Proposition (IS 2016)

Explicit description of the block diagonal below the main diagonal.

$$
\begin{array}{c c}
 & \begin{array}{cccc} \overbrace{\qquad\quad}^{\beta \vdash n} & \overbrace{\qquad\quad}^{\beta \vdash n-1} & \cdots & \overbrace{\quad}^{\beta \vdash 0} \end{array} \\
\begin{array}{l} \alpha \vdash n \left\{ \begin{array}{c} \\ \\ \end{array} \right. \\ \alpha \vdash n-1 \left\{ \begin{array}{c} \\ \\ \end{array} \right. \\ \vdots \qquad \vdots \\ \alpha \vdash 0 \quad \{ \end{array} &
\left(\begin{array}{cc|cc|cc|c|c}
 & & 0 & 0 & 0 & 0 & \cdots & 0 \\
 & I & & & & & & \\
 & & 0 & 0 & 0 & 0 & & 0 \\
\hline
\checkmark & \checkmark & \ddots & & 0 & 0 & & 0 \\
\checkmark & \checkmark & & & 0 & 0 & & 0 \\
\hline
* & * & \checkmark & & \ddots & & & \\
\hline
\vdots & & & & \checkmark & & \ddots & \\
\hline
* & * & & & & & \checkmark & I
\end{array}\right)
\end{array}
$$

Cartan matrix

Proposition (IS 2016)

Let $\alpha \vdash k$ and $\beta \vdash k+1$. The number of times that S_α appears as a J-H factor in P_β is the number of times that S^β appears as an irreducible constituent in

$$\operatorname{Ind}_{S_{k-1}\times S_2}^{S_{k+1}}(\operatorname{Res}_{S_{k-1}}^{S_k} S^\alpha \otimes \operatorname{tr}_{S_2})$$

which is the number of ways to obtain β from α by removing one box and adding two but not in the same column.

Cartan matrix

Proposition (IS)

Explicit description of another block diagonal.

$$
\begin{array}{r l}
 & \qquad \overbrace{\qquad\qquad}^{\beta \vdash n} \qquad \overbrace{\qquad\qquad}^{\beta \vdash n-1} \qquad \cdots \qquad \overbrace{\quad}^{\beta \vdash 0} \\
\begin{array}{r}
\alpha \vdash n \left\{ \begin{array}{c} \\ \\ \end{array} \right. \\
\alpha \vdash n-1 \left\{ \begin{array}{c} \\ \\ \end{array} \right. \\
\\
\vdots \qquad \vdots \\
\alpha \vdash 0 \quad \{
\end{array}
&
\left(
\begin{array}{cc|cc|cc|c|c}
 & & 0 & 0 & 0 & 0 & \cdots & 0 \\
 & I & & & & & & \\
 & & 0 & 0 & 0 & 0 & & 0 \\
\hline
\checkmark & \checkmark & \ddots & & 0 & 0 & & 0 \\
\checkmark & \checkmark & & & 0 & 0 & & 0 \\
\hline
\checkmark & \checkmark & \checkmark & & \ddots & & & \\
\hline
\vdots & & & & \checkmark & & \ddots & \\
\hline
* & * & & & \checkmark & & \checkmark & I
\end{array}
\right)
\end{array}
$$

Cartan matrix

Proposition (IS)

Let $\alpha \vdash k$ and $\beta \vdash k+2$. The number of times that S_α appears as a J-H factor in P_β is the number of times that S^β appears as an irreducible constituent in

$$\operatorname{Ind}_{S_{k-1}\times S_3}^{S_{k+2}}(\operatorname{Res}_{S_{k-1}}^{S_k}(S^\alpha)\otimes \operatorname{tr}_{S_3}) \oplus \operatorname{Ind}_{S_{k-2}\times D_4}^{S_{k+2}} \overline{\operatorname{Res}_{S_{k-2}\times S_2}^{S_k} S^\alpha}.$$

The projective module of the "dual standard" partition

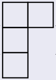
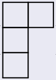
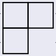
Conjecture (Walter Mazorchuk)

Consider the projective indecomposable module P_β for the partition $\beta = [2, 1^{n-2}]$. It contains only few J-H components.

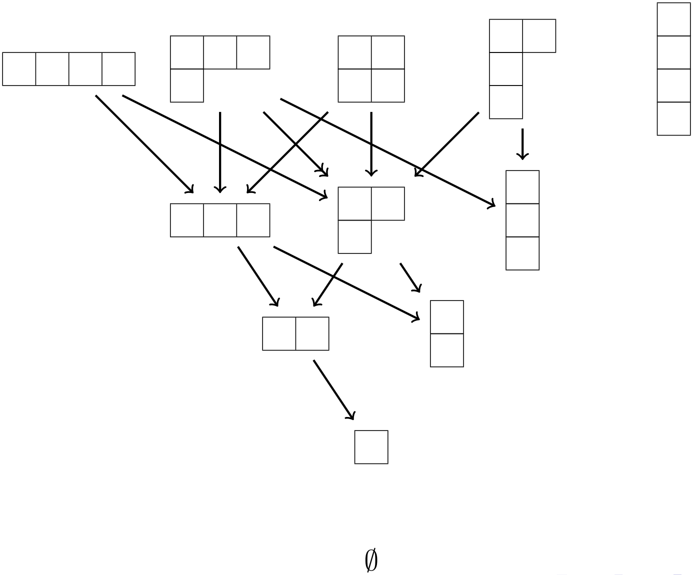
Proposition (IS)

For $n \geq 3$, the only J-H components of P_β are the simples for $[2, 1^{n-2}]$, $[2, 1^{n-3}]$ and $[1^{n-1}]$. Each one with multiplicity 1.

Example ($n = 4$)

The J-H components of $P($$)$ are $S($$)$, $S($$)$ and $S($$)$.

Quiver of $\mathbb{C}\mathcal{PT}_4$



Homological arguments-example

- Consider the short exact sequence

$$0 \to \mathrm{Rad}\, P(\ydiagram{2,1,1}) \to P(\ydiagram{2,1,1}) \to S(\ydiagram{2,1,1}) \to 0$$

- By the above we know that the J-H components of Rad $P(\ydiagram{2,1,1})$ are $S(\ydiagram{2,1})$ and $S(\ydiagram{1,1,1})$.
- Other known facts:
  - $S(\ydiagram{1,1,1})$is a projective module.
  - $\mathrm{Ext}^1(S(\ydiagram{2,1}), S(\ydiagram{1,1,1})) = \mathrm{Ext}^1(S(\ydiagram{1,1,1}), S(\ydiagram{2,1})) = 0$

    This implies that Rad $P(\ydiagram{2,1,1}) = S(\ydiagram{2,1}) \oplus S(\ydiagram{1,1,1})$.

Homological arguments-example

- Consider the short exact sequence

$$0 \to S(\yng(2,1)) \oplus S(\yng(1,1)) \to P(\yng(2,1,1)) \to S(\yng(2,1,1)) \to 0$$

- By the "long exact sequence" Theorem we have that

$$\operatorname{Ext}^{k}(S(\yng(2,1,1)), S(\yng(1))) \cong \operatorname{Ext}^{k-1}(S(\yng(2,1)) \oplus S(\yng(1,1)), S(\yng(1)))$$

$$\operatorname{Ext}^{k-1}(S(\yng(2,1)) \oplus S(\yng(1,1)), S(\yng(1))) =$$

$$\operatorname{Ext}^{k-1}(S(\yng(2,1)), S(\yng(1))) \oplus \operatorname{Ext}^{k-1}(S(\yng(1,1)), S(\yng(1))) =$$

$$\operatorname{Ext}^{k-1}(S(\yng(2,1)), S(\yng(1)))$$

- Hence

$$\operatorname{Ext}^{k}(S(\yng(2,1,1)), S(\yng(1))) \cong \operatorname{Ext}^{k-1}(S(\yng(2,1)), S(\yng(1)))$$

Homological arguments

- Consider the short exact sequence

$$0 \to \operatorname{Rad} P([2,1^{n-2}]) \to P([2,1^{n-2}]) \to S([2,1^{n-2}]) \to 0$$

- By the above we know that the J-H components of $\operatorname{Rad} P([2,1^{n-2}])$ are $S([2,1^{n-3}])$ and $S([1^{n-1}])$.
- Other known facts:
  - $S([1^{n-1}])$ is a projective module.
  - $\operatorname{Ext}^1(S([2,1^{n-3}]),S([1^{n-1}])) = \operatorname{Ext}^1(S([1^{n-1}]),S([2,1^{n-3}])) = 0$
    This implies that

$$\operatorname{Rad} P([2,1^{n-2}]) = S([2,1^{n-3}]) \oplus S([1^{n-1}])$$

Homological arguments

- Consider the short exact sequence

$$0 \to S([2,1^{n-3}]) \oplus S([1^{n-1}]) \to P([2,1^{n-2}]) \to S([2,1^{n-2}]) \to 0$$

- By the "long exact sequence" Theorem we have that

$$\mathrm{Ext}^k(S([2,1^{n-2}]), S([1])) \cong \mathrm{Ext}^{k-1}(S([2,1^{n-3}]) \oplus S([1^{n-1}]), S([1]))$$

$$\begin{aligned}\mathrm{Ext}^{k-1}(S([2,1^{n-3}]) \oplus S([1^{n-1}]), S([1])) &= \mathrm{Ext}^{k-1}(S([2,1^{n-3}]), S([1])) \\ &\oplus \mathrm{Ext}^{k-1}(S([1^{n-1}]), S([1])) = \\ &\mathrm{Ext}^{k-1}(S([2,1^{n-3}]), S([1]))\end{aligned}$$

- Hence

$$\mathrm{Ext}^k(S([2,1^{n-2}]), S([1])) \cong \mathrm{Ext}^{k-1}(S([2,1^{n-3}]), S([1]))$$

Homological arguments

- This implies that

$$\operatorname{pd}(S([2,1^{n-2}])) = \operatorname{pd}(S([2,1^{n-2}])) + 1$$

- So by easy induction we obtain

$$\operatorname{gl.dim}(\mathbb{C}\mathcal{PT}_n) = n - 1$$

as required.

Thank you!