Lattice-valued functions

Eszter K. Horváth, Szeged

Co-authors: Branimir Šešelja, Andreja Tepavčević

Novi Sad, 2017, june 15.
Let $S$ be a nonempty set and $L$ a complete lattice. Every mapping $\mu : S \to L$ is called a **lattice-valued (L-valued) function** on $S$. 
Let $p \in L$. A **cut set** of an $L$-valued function $\mu : S \to L$ (a $p$-cut) is a subset $\mu_p \subseteq S$ defined by:

$$x \in \mu_p \text{ if and only if } \mu(x) \geq p.$$  

(1)

In other words, a $p$-cut of $\mu : S \to L$ is the inverse image of the principal filter $\uparrow p$, generated by $p \in L$:

$$\mu_p = \mu^{-1}(\uparrow p).$$

(2)

It is obvious that for every $p, q \in L$, $p \leq q$ implies $\mu_q \subseteq \mu_p$. 

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In other words, a $p$-cut of $\mu : S \to L$ is the inverse image of the principal filter $\uparrow p$, generated by $p \in L$:

$$\mu_p = \mu^{-1}(\uparrow p). \quad (2)$$

It is obvious that for every $p, q \in L$, $p \leq q$ implies $\mu_q \subseteq \mu_p$. 
Let \( p \in L \). A **cut set** of an \( L \)-valued function \( \mu : S \to L \) (a \( p \)-cut) is a subset \( \mu_p \subseteq S \) defined by:

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\forall x \in \mu_p \text{ if and only if } \mu(x) \geq p.
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In other words, a \( p \)-cut of \( \mu : S \to L \) is the inverse image of the principal filter \( \uparrow p \), generated by \( p \in L \):

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It is obvious that for every \( p, q \in L \), \( p \leq q \) implies \( \mu_q \subseteq \mu_p \).
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Cuts and closure systems

If $\mu : S \rightarrow L$ is an $L$-valued function on $S$, then the collection $\mu_L$ of all cuts of $\mu$ is a closure system on $S$ under the set-inclusion.

Let $\mathcal{F}$ be a closure system on a set $S$. Then there is a lattice $L$ and an $L$-valued function $\mu : S \rightarrow L$, such that the collection $\mu_L$ of cuts of $\mu$ is $\mathcal{F}$.

A required lattice $L$ is the collection $\mathcal{F}$ ordered by the reversed-inclusion, and that $\mu : S \rightarrow L$ can be defined as follows:

$$\mu(x) = \bigcap \{ f \in \mathcal{F} \mid x \in f \}. \quad (3)$$
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The relation $\approx$ on $L$

Given an $L$-valued function $\mu : S \to L$, we define the relation $\approx$ on $L$: for $p, q \in L$

$$p \approx q \text{ if and only if } \mu_p = \mu_q.$$  \hfill (4)

The relation $\approx$ is an equivalence on $L$, and

$$p \approx q \text{ if and only if } \uparrow p \cap \mu(S) = \uparrow q \cap \mu(S),$$ \hfill (5)

where $\mu(S) = \{ r \in L \mid r = \mu(x) \text{ for some } x \in S \}$.

We denote by $L/\approx$ the collection of equivalence classes under $\approx$. 
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The collection of cuts

Let $(\mu_L, \leq)$ be the poset with $\mu_L = \{\mu_p \mid p \in L\}$ (the collection of cuts of $\mu$) and the order $\leq$ being the inverse of the set-inclusion: for $\mu_p, \mu_q \in \mu_L$,

$$\mu_p \leq \mu_q \text{ if and only if } \mu_q \subseteq \mu_p.$$  

$(\mu_L, \leq)$ is a complete lattice and for every collection $\{\mu_p \mid p \in L_1\}$, $L_1 \subseteq L$ of cuts of $\mu$, we have

$$\bigcap\{\mu_p \mid p \in L_1\} = \mu_{\lor\{p \mid p \in L_1\}}.$$  

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Let \((\mu_L, \leq)\) be the poset with \(\mu_L = \{\mu_p \mid p \in L\}\) (the collection of cuts of \(\mu\)) and the order \(\leq\) being the inverse of the set-inclusion: for \(\mu_p, \mu_q \in \mu_L\),

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\bigcap\{\mu_p \mid p \in L_1\} = \mu \lor (p \mid p \in L_1). \tag{6}
\]
The quotient $L/\approx$

Each $\approx$-class contains its supremum:

$$\bigvee [p]_\approx \in [p]_\approx.$$  \hfill (7)

The mapping $p \mapsto \bigvee [p]_\approx$ is a closure operator on $L$.

The quotient $L/\approx$ can be ordered by the relation $\leq_{L/\approx}$ defined as follows:

$$[p]_\approx \leq_{L/\approx} [q]_\approx \text{ if and only if } \uparrow q \cap \mu(S) \subseteq \uparrow p \cap \mu(S).$$

The order $\leq_{L/\approx}$ of classes in $L/\approx$ corresponds to the order of suprema of classes in $L$ (we denote the order in $L$ by $\leq_L$):
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The poset \((L/\approx, \leq_{L/\approx})\)

\(\text{The poset } (L/\approx, \leq_{L/\approx}) \text{ is a complete lattice fulfilling:}\)

1. \([p]_\approx \leq_{L/\approx} [q]_\approx \text{ if and only if } \bigvee[p]_\approx \leq_{L} \bigvee[q]_\approx.\)
2. The mapping \([p]_\approx \mapsto \bigvee[p]_\approx\) is an injection of \(L/\approx\) into \(L.\)

The sub-poset \((\bigvee[p]_\approx, \leq_L)\) of \(L\) is isomorphic to the lattice \((L/\approx, \leq_{L/\approx})\) under \(\bigvee[p]_\approx \mapsto [p]_\approx.\)

Let \(\mu : S \to L\) be an \(L\)-valued function on \(S.\) The lattice \((\mu_L, \leq)\) of cuts of \(\mu\) is isomorphic with the lattice \((L/\approx, \leq_{L/\approx})\) of \(\approx\)-classes in \(L\) under the mapping \(\mu_p \mapsto [p]_\approx.\)
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We take the lattice \((\mathcal{F}, \leq)\), where \(\mathcal{F} = \mu_L \subseteq \mathcal{P}(S)\) is the collection of cuts of \(\mu\), and the order \(\leq\) is the dual of the set inclusion.

Let \(\hat{\mu} : S \to \mathcal{F}\), where

\[
\hat{\mu}(x) := \bigcap \{\mu_p \in \mu_L \mid x \in \mu_p\}.
\] (8)

Properties:
\(\hat{\mu}\) has the same cuts as \(\mu\).
\(\hat{\mu}\) has one-element classes of the equivalence relation \(\approx\).
Every \(f \in \mathcal{F}\) is equal to the corresponding cut of \(\hat{\mu}\).
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Example

\[ S = \{a, b, c, d\} \]

\[ \mu = \begin{pmatrix} a & b & c & d \\ p & s & r & t \end{pmatrix} \quad \nu = \begin{pmatrix} a & b & c & d \\ z & w & m & v \end{pmatrix} \]

\[ \hat{\mu} = \hat{\nu} = \begin{pmatrix} a & b & c & d \\ \{a\} & \{a, b\} & \{c\} & \{c, d\} \end{pmatrix} \]
A **Boolean function** is a mapping $f : \{0, 1\}^n \rightarrow \{0, 1\}$, $n \in \mathbb{N}$.

A **lattice-valued Boolean function** is a mapping

$$f : \{0, 1\}^n \rightarrow L,$$

where $L$ is a complete lattice.

We also deal with **lattice-valued $n$-variable functions** on a finite domain $\{0, 1, \ldots, k - 1\}$:

$$f : \{0, 1, \ldots, k - 1\}^n \rightarrow L,$$

where $L$ is a complete lattice.

We use also **$p$-cuts** of lattice-valued functions as characteristic functions: for $f : \{0, 1, \ldots, k - 1\}^n \rightarrow L$ and $p \in L$, we have

$$f_p : \{0, 1, \ldots, k - 1\}^n \rightarrow \{0, 1\},$$

such that $f_p(x_1, \ldots, x_n) = 1$ if and only if $f(x_1, \ldots, x_n) \geq p$.

Clearly, a cut of a lattice-valued Boolean function is (as a characteristic function) a Boolean function.
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Clearly, a cut of a lattice-valued Boolean function is (as a characteristic function) a Boolean function.
As usual, by $S_n$ we denote the symmetric group of all permutations over an $n$-element set. If $f$ is an $n$-variable function on a finite domain $X$ and $\sigma \in S_n$, then $f$ is invariant under $\sigma$, symbolically $\sigma \vdash f$, if for all $(x_1, \ldots, x_n) \in X^n$

$$f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) .$$

If $f$ is invariant under all permutations in $G \leq S_n$ and not invariant under any permutation from $S_n \setminus G$, then $G$ is called the invariance group of $f$, and it is denoted by $G(f)$.
Invariance group

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A group $G \leq S_n$ is said to be $(k, m)$-representable if there is a function $f : \{0, 1, \ldots, k-1\}^n \to \{1, \ldots, m\}$ whose invariance group is $G$.

If $G$ is the invariance group of a function $f : \{0, 1, \ldots, k-1\}^n \to \mathbb{N}$, then it is called $(k, \infty)$-representable.

$G \leq S_n$ is called $m$-representable if it is the invariance group of a function $f : \{0, 1\}^n \to \{1, \ldots, m\}$; it is called representable if it is $m$-representable for some $m \in \mathbb{N}$.

By the above, representability is equivalent with $(2, \infty)$-representability.
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In particular, a $(2, L)$-representable group is the invariance group of a lattice-valued Boolean function $f : \{0, 1\}^n \to L$.

The notion of $(2, L)$-representability is more general than $(2, 2)$-representability. An example is the Klein 4-group: 
\{id, (12)(34), (13)(24), (14)(23)\}, which is $(2, L)$ representable (for $L$ being a three element chain), but not $(2, 2)$-representable.
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Let $O_k^{(n)} = \{ f \mid f : k^n \to k \}$ denote the set of all $n$-ary operations on $k$, and for $F \subseteq O_k^{(n)}$ and $G \subseteq S_n$ let

$$F^{\bot} := \{ \sigma \in S_n \mid \forall f \in F : \sigma \vdash f \}, \quad \overline{F}^{(k)} := (F^{\bot})^{\bot},$$

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The assignment $G \mapsto \overline{G}^{(k)}$ is a closure operator on $S_n$, and it is easy to see that $\overline{G}^{(k)}$ is a subgroup of $S_n$ for every subset $G \subseteq S_n$ (even if $G$ is not a group). For $G \leq S_n$, we call $\overline{G}^{(k)}$ the Galois closure of $G$ over $k$, and we say that $G$ is Galois closed over $k$ if $\overline{G}^{(k)} = G$. 
A Galois connection for invariance groups

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Eszter K. Horváth, Szeged

Co-authors: Branimir ˇSeˇselja, Andreja Tepavˇcevi´c ()

Lattice-valued functions

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A group $G \leq S_n$ is Galois closed over $k$ if and only if $G$ is $(k, \infty)$-representable.

For every $G \leq S_n$, we have

$$\overline{G^{(k)}} = \bigcap_{a \in k^n} (S_n)_a \cdot G.$$ 

For arbitrary $k, n \geq 2$, characterize those subgroups of $S_n$ that are Galois closed over $k$.

Theorem (H., Makay, Pöschel, Waldhauser) Let $n > \max (2^d, d^2 + d)$ and $G \leq S_n$. Then $G$ is not Galois closed over $k$ if and only if $G = A_B \times L$ or $G <_{sd} S_B \times L$, where $B \subseteq n$ is such that $D := n \setminus B$ has less than $d$ elements, and $L$ is an arbitrary permutation group on $D$. 

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Eszter K. Horváth, Szeged Co-authors: Branimir ˇSeˇselja, Andreja Tepavˇcevi´c ()

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One can easily check that a permutation group \( G \subseteq S_n \) is \( L \)-representable if and only if it is Galois closed over \( 2 \).

Similarly, it is easy to show that a permutation group is \((k, L)\)-representable if and only if it is Galois closed over the \( k \)-element domain.
One can easily check that a permutation group $G \subseteq S_n$ is $L$-representable if and only if it is Galois closed over $2$.

Similarly, it is easy to show that a permutation group is $(k, L)$-representable if and only if it is Galois closed over the $k$-element domain.
**Theorem** Let $L$ be a complete lattice, let $A \neq \emptyset$ be a set and let $\sigma : A \to A$, $\mu : A \to L$, $\psi : L \to L$. Then, for every $p \in L$,

$$
(\sigma \circ \mu \circ \psi)_p = \sigma \circ \mu \circ \psi \circ p.
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**Corollary** Let $L$ be a complete lattice, let $A \neq \emptyset$ and let $\mu : A \to L$. Then the following holds.

(i) $\mu_p = \mu \circ (I_L)_p$, where $I_L$ is the identity mapping $I_L : L \to L$.

(ii) $(\sigma \circ \mu)_p = \sigma \circ \mu \circ p$, for $\sigma : A \to A$.

(iii) $(\mu \circ \psi)_p = \mu \circ \psi \circ p$, where $\psi$ is a map $\psi : L \to L$. 
Cuts of composition of functions

**Theorem** Let $L$ be a complete lattice, let $A \neq \emptyset$ be a set and let $\sigma : A \to A$, $\mu : A \to L$, $\psi : L \to L$. Then, for every $p \in L$,

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**Theorem** Let $L$ be a complete lattice, let $A \neq \emptyset$ be a set and let
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**Theorem** Let \( L \) be a complete lattice, let \( A \neq \emptyset \) be a set and let \( \sigma : A \rightarrow A, \mu : A \rightarrow L, \psi : L \rightarrow L \). Then, for every \( p \in L \),

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\begin{enumerate}
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\end{enumerate}
Proposition Let $f : \{0, \ldots, k - 1\}^n \to L$ and $\sigma \in S_n$. Then

$$\sigma \vdash f \text{ if and only if for every } p \in L, \sigma \vdash f_p.$$ 

The invariance group of a lattice-valued function $f$ depends only on the canonical representation of $f$.

If $f_1 : \{0, \ldots, k - 1\}^n \to L_1$ and $f_2 : \{0, \ldots, k - 1\}^n \to L_2$ are two $n$-variable lattice-valued functions on the same domain, then $\hat{f}_1 = \hat{f}_2$ implies $G(f_1) = G(f_2)$. 
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For every $n \in \mathbb{N}$, there is a lattice $L$ and a lattice valued Boolean function $F : \{0, 1\}^n \rightarrow L$ satisfying the following: If $G \leq S_n$ and $G = G(f)$ for a Boolean function $f$, then $G = G(F_p)$, for a cut $F_p$. 
Representation theorem on the $k$-element set

Every subgroups of $S_n$ is an invariance group of a function
\[\{0, \ldots, k - 1\}^n \to \{0, \ldots, k - 1\}\] if and only if $k \geq n$.

If $k \geq n$, then for every subgroup $G$ of $S_n$ there exists a function $f : \{0, \ldots, k - 1\}^n \to \{0, 1\}$ such that the invariance group of $f$ is exactly $G$.

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A lattice-valued Boolean function is a map $\mu: \{0, 1\}^n \to L$ where $L$ is a bounded lattice and $n \in \langle 1, 2, 3, \ldots \rangle$.

We say that $\mu$ can be given by a linear combination (in $L$) if there are $w_1, \ldots, w_n \in L$ such that, for all $x = \{x_1, \ldots, x_n\} \in \{0, 1\}^n$,

$$\mu(x) = \bigvee_{i=1}^n w_i x_i, \quad \text{that is,} \quad \mu(x) = \bigvee_{i=1}^n (w_i \land x_i).$$

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Cuts and closure systems

For $p \in L$, the set

$$\mu_p := \{ x \in \{0, 1\}^n : \mu(x) \geq p \}$$  \hspace{1cm} (10)

is called a cut of $\mu$.

A closure system $\mathcal{F}$ over $B_n$ is a $\cap$-subsemilattice of the powerset lattice $P(B_n) = \langle P(B_n); \cup, \cap \rangle$ such that $B_n \in \mathcal{F}$. By finiteness, $\mathcal{F}$ is necessarily a complete $\cap$-semilattice.

A closure system $\mathcal{F}$ determines a closure operator in the standard way. We only need the closures of singleton sets, that is,

$$\text{for } x \in B_n, \text{ we have } \overline{x} := \bigcap \{ f \in \mathcal{F} : x \in f \}. \hspace{1cm} (11)$$
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If $\mu : B_n \rightarrow L$ such that $\mu(0) = 0$ and, for all $x, y \in B_n$, 
$\mu(x \lor y) = \mu(x) \lor \mu(y)$, then $\mu$ is called a \(\{\lor, 0\}\)-homomorphism.

A lattice-valued function $B_n \rightarrow L$ can be given by a linear combination in $L$ iff it is a \(\{\lor, 0\}\)-homomorphism.

$$\mu(x \lor y) = \bigvee_i w_i (x_i \lor y_i) = \bigvee_i (w_i x_i \lor w_i y_i) = \bigvee_i w_i x_i \lor \bigvee_i w_i y_i = \mu(x) \lor \mu(y).$$

Let $e^{(i)} = \langle 0, \ldots, 0, 1, 0, \ldots, 0 \rangle \in B_n$ where 1 stands in the $i$-th place. Define $w_i := \mu(e^{(i)})$. Observe that $\mu(e^{(i)} \cdot 1) = w_i = w_i \cdot 1$ and $\mu(e^{(i)} \cdot 0) = 0 = w_i \cdot 0$, that is, $\mu(e^{(i)} \cdot x_i) = w_i \cdot x_i$. Therefore, for $x \in B_n$, we obtain $\mu(x) = \mu(\bigvee_i e^{(i)} x_i) = \bigvee_i \mu(e^{(i)} x_i) = \bigvee_i w_i \cdot x_i$. 
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A lattice-valued function $B_n \to L$ can be given by a linear combination in $L$ iff it is a $\{\lor, 0\}$-homomorphism.

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Let $e^{(i)} = \langle 0, \ldots, 0, 1, 0, \ldots, 0 \rangle \in B_n$ where 1 stands in the $i$-th place. Define $w_i := \mu(e^{(i)})$. Observe that $\mu(e^{(i)} \cdot 1) = w_i = w_i \cdot 1$ and $\mu(e^{(i)} \cdot 0) = 0 = w_i \cdot 0$, that is, $\mu(e^{(i)} \cdot x_i) = w_i \cdot x_i$. Therefore, for $x \in B_n$, we obtain

$$\mu(x) = \mu(\bigvee_i e^{(i)} x_i) = \bigvee_i \mu(e^{(i)} x_i) = \bigvee_i w_i \cdot x_i.$$
If $\emptyset \neq X \subseteq B_n$ such that $(\forall x \in X)(\forall y \in B_n)(x \leq y \quad \text{then} \quad y \in X)$, then $X$ is an *up-set* of $B_n$.

The lattice-valued function $\mu: B_n \to L$ is isotone iff all the cuts of $\mu$ are up-sets.
Up-sets

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The lattice-valued function $\mu : B_n \to L$ is isotone iff all the cuts of $\mu$ are up-sets.
Let $\mathcal{F}$ be a set consisting of some up-sets of $B_n$. Then, the following three conditions are equivalent.

(i) $\mathcal{F}$ is a closure system over $B_n$, and for all $x, y \in B_n$, $x \subseteq y$ implies $x \lor y = x$.

(ii) $\mathcal{F}$ is a closure system over $B_n$, and for all $x, y \in B_n$, $x \lor y = x \cap y$.

(iii) There exist a bounded lattice $L$ and a lattice-valued function $\mu : B_n \to L$ given by a linear combination such that $\mathcal{F}$ is the family of cuts of $\mu$. 
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