# Systems of two-sided linear fuzzy relation equations and inequalities and their applications

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# Fuzzy relation equations and inequalities (FREIs)

## Origins of FRE

- ⋆ ***E. Sanchez*** (1974) – the first who dealt with FREI's and systems of FREI's;
- ⋆ he used them in *medical research*;
- ⋆ they were first studied over the Gödel structure

  later, more general structures of truth values were used, including *complete residuated lattices*

## Applications

- ⋆ *fuzzy control;*
- ⋆ *discrete dynamic systems*;
- ⋆ *knowledge engineering*;
- ⋆ *decision-making*;
- ⋆ *fuzzy information retrieval*;
- ⋆ *fuzzy pattern recognition*;
- ⋆ *image compression and reconstruction*;
- ⋆ *fuzzy automata theory*;
- ⋆ *fuzzy social network analysis*, etc.

# Fuzzy relations

## Fuzzy sets and fuzzy relations

- structure of truth values – ***complete residuated lattice*** $\mathbb{L} = (L, \wedge, \vee, \otimes, \to, 0, 1)$

  ***residuation property*** for $\otimes$ and $\to$: $x \otimes y \leqslant z \;\Leftrightarrow\; x \leqslant y \to z$
- ***fuzzy subsets*** of $A$ – functions of $A$ to $L$, $L^A$ – all fuzzy subsets of $A$
- ***fuzzy relations*** between $A$ and $B$ – functions of $A \times B$ to $L$

  $L^{A\times B}$ – all fuzzy relations between $A$ and $B$, $L^{A\times A}$ – fuzzy relations on $A$
- ***inclusion***: $f \leqslant g \;\Leftrightarrow\; f(x) \leqslant g(x)$, for every $x \in A$
- ***intersection*** and ***union***

$$\Big(\bigwedge_{i\in I} f_i\Big)(x) = \bigwedge_{i\in I} f_i(x), \qquad \Big(\bigvee_{i\in I} f_i\Big)(x) = \bigvee_{i\in I} f_i(x)$$

- for $R \in L^{A\times B}$, the ***inverse fuzzy relation*** $R^{-1} \in L^{B\times A}$ is given by $R^{-1}(b,a) = R(a,b)$
- ***crisp subsets*** – take only values 0 and 1, $2^A$ – all crisp subsets of $A$
- if $A$ and $B$ are finite with $|A| = m$ and $|B| = n$:

  *fuzzy subsets* of $A$ $\equiv$ *fuzzy vectors* of length $m$ with entries in $\mathbb{L}$

  *fuzzy relations* between $A$ and $B$ $\equiv$ *fuzzy matrices* of type $m \times n$ with entries in $\mathbb{L}$

# Fuzzy relational compositions

## Compositions of fuzzy relations and fuzzy sets

- ***Fuzzy relational composition*** – $R \in L^{A\times B}$, $S \in L^{B\times C}$, $R \circ S \in L^{A\times C}$

$$(R \circ S)(a,c) = \bigvee_{b\in B} R(a,b) \otimes S(b,c)$$

***matrix product***

- ***Set-relation compositions*** – $f \in L^A$, $R \in L^{A\times B}$, $g \in L^B$, $f \circ R \in L^B$, $R \circ g \in L^A$

$$(f \circ R)(b) = \bigvee_{a\in A} f(a) \otimes R(a,b), \qquad (R \circ g)(a) = \bigvee_{b\in B} R(a,b) \otimes g(b)$$

***vector-matrix products***

- ***Set-set composition*** – $f, g \in L^A$, $f \circ g \in L$ (scalar)

$$f \circ g = \bigvee_{a\in A} f(a) \otimes g(a)$$

***scalar product*** or ***dot product***

# Residuals of fuzzy relations

**Residuals of fuzzy relations $R \in L^{A\times B}$, $S \in L^{B\times C}$ and $T \in L^{A\times C}$**

- ***right residual of** $T$ **by** $R$* – $R\backslash T \in L^{B\times C}$

$$(R\backslash T)(b,c) = \bigwedge_{a\in A} R(a,b) \to T(a,c)$$

- ***left residual of** $T$ **by** $S$* – $T/S \in L^{A\times B}$

$$(T/S)(a,b) = \bigwedge_{c\in C} S(b,c) \to T(a,c)$$

- ***residuation property*** – $R \circ S \leqslant T \quad \Leftrightarrow \quad S \leqslant R\backslash T \quad \Leftrightarrow \quad R \leqslant T/S$

**Residuals of fuzzy sets $f \in L^A$ and $g \in L^B$**

- ***right residual of** $g$ **by** $f$* – $f\backslash g \in L^{A\times B}$ – $(f\backslash g)(a,b) = f(a) \to g(b)$
- ***left residual of** $g$ **by** $f$* – $g/f \in L^{B\times A}$ – $(g/fg)(b,a) = f(a) \to g(b)$ $[g/f = (f\backslash g)^{-1}]$
- ***residuation property*** – $f \circ R \leqslant g \;\Leftrightarrow\; R \leqslant f\backslash g, \qquad S \circ f \leqslant g \;\Leftrightarrow\; S \leqslant g/f$

**Other types of residuals**

Fuzzy relation by fuzzy set, fuzzy set by fuzzy relation, residuals including scalars, etc.

# Special types of fuzzy relations

## Fuzzy quasi-orders and fuzzy equivalences

- ⋆ $R \in L^{A\times A}$ is
  - ***reflexive*** if $R(a,a) = 1$, for all $a \in A$
  - ***symmetric*** if $R(a,b) = R(b,a)$, for all $a,b \in A$
  - ***transitive*** if $R(a,b) \otimes R(b,c) \leqslant R(a,c)$, for all $a,b,c \in A$
- ⋆ ***fuzzy quasi-order*** – reflexive and transitive fuzzy relation
- ⋆ ***fuzzy equivalence*** – reflexive, symmetric and transitive fuzzy relation

# Fuzzy relational systems

**One-mode fuzzy relational system – $\mathscr{O} = (A, \mathscr{R})$**

- $A$ – non-empty set
- $\mathscr{R} = \{R_i\}_{i\in I} \subset L^{A\times A}$

**Two-mode fuzzy relational systems – $\mathscr{T} = (A, B, \mathscr{R})$**

- $A, B$ – two different non-empty sets
- $\mathscr{R} = \{R_i\}_{i\in I} \subset L^{A\times B}$

**Multi-mode fuzzy relational systems – $\mathscr{M} = (A_1, \ldots, A_n, \mathscr{R})$**

- $A_1, \ldots, A_n$ – non-empty sets
- $\mathscr{R}$ – system of fuzzy relations between $A_j$ and $A_k$ defined for some pairs $(j, k)$
- formally:
  - $J \subseteq [1, n] \times [1, n]$ such that $(\forall j \in [1, n])(\exists k \in [1, n])\ (j, k) \in J$ or $(k, j) \in J$
  - $\{I_{j,k}\}_{(j,k)\in J}$ – collection of non-empty sets
  - $\mathscr{R} = \{R_i^{j,k} \mid (j, k) \in J,\ i \in I_{j,k}\}$
  - $R_i^{j,k} \in L^{A_j\times A_k}$, for all $(j, k) \in J$ and $i \in I_{j,k}$
- complex synthesis of one-mode and two-mode fuzzy relational systems

# Systems of linear FREIs

## Linear FREIs

- ★ ***Sanchez*** (1974) – ***linear FREIs***
- ★ one side – *linear function of an unknown*, another side – a *constant* (fixed fuzzy relation)
- ★ general form: $R \circ \alpha \circ S = T$, $R \circ \alpha \circ S \leqslant T$ or $R \circ \alpha \circ S \geqslant T$, where

  $R \in L^{A\times B}$, $S \in L^{C\times D}$, $T \in L^{A\times D}$

  $\alpha$ – unknown taking values in $L^{B\times C}$
- ★ some $R$, $S$, $T$ and $\alpha$ also may be fuzzy sets
- ★ mostly studied: $R \circ \alpha = S$, where

  $\alpha$ is an unknown fuzzy relation and $S$ is a given fuzzy relation, or

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## Solvability of systems of linear FREIs

- ★ ***R. A. Cuninghame-Green, K. Cechlárová*** (1995)
  - $\bigwedge_{i\in I} R_i\backslash T_i/S_i$ *is the greatest solution of the system* $R_i \circ \alpha \circ S_i \leqslant T_i,\ i \in I$
  - *if* $\bigwedge_{i\in I} R_i\backslash T_i/S_i$ *is a solution of the system* $R_i \circ \alpha \circ S_i = T_i,\ i \in I$, *then it is its greatest solution*

    *otherwise, the system* $R_i \circ \alpha \circ S_i = T_i,\ i \in I$, *does not have any solution*

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# Systems of weakly linear FREIs

## Two-sided linear FREIs

- both sides are *linear functions of one unknown or two different unknowns*

## Homogeneous systems of weakly linear FREIs

- $\mathscr{O} = (A, \{R_i\}_{i\in I})$ – one-mode fuzzy relational system
- Homogeneous systems of weakly linear FREIs are

$$\alpha \circ R_i \leqslant R_i \circ \alpha, \quad i \in I$$
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$$\alpha \circ R_i = R_i \circ \alpha, \quad i \in I$$

$\alpha$ is an unknown fuzzy relation on the set $A$

## Homogeneous systems of weakly linear FREIs

- $\mathscr{O} = (A, \{R_i\}_{i\in I})$, $\mathscr{O}' = (A', \{R'_i\}_{i\in I})$ – two one-mode fuzzy relational systems
- Heterogeneous systems of weakly linear FREIs are

$$\alpha \circ R'_i \leqslant R_i \circ \alpha, \quad i \in I$$
$$\alpha \circ R'_i \geqslant R_i \circ \alpha, \quad i \in I$$
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# Greatest solutions of weakly linear systems

## Equivalent forms of weakly linear systems

- using residuals we obtain equivalent forms of weakly linear systems

$$\alpha \leqslant \bigwedge_{i\in I} (R_i \circ \alpha)/R_i \qquad \alpha \leqslant \bigwedge_{i\in I} (R_i \circ \alpha)/R'_i$$

$$\alpha \leqslant \bigwedge_{i\in I} R_i\backslash(\alpha \circ R_i) \qquad \alpha \leqslant \bigwedge_{i\in I} R_i\backslash(\alpha \circ R'_i)$$

$$\alpha \leqslant \bigwedge_{i\in I} (R_i \circ \alpha)/R_i \wedge R_i\backslash(\alpha \circ R_i) \qquad \alpha \leqslant \bigwedge_{i\in I} (R_i \circ \alpha)/R'_i \wedge R_i\backslash(\alpha \circ R'_i)$$

## Greatest solutions of weakly linear systems

- expressions on the right-hand sides of these inequalities define isotone functions on $\alpha$
- solutions of the inequalities are post-fixed points of these functions – $\alpha \leqslant \phi(\alpha)$
- ***Knaster-Tarski Fixed Point Theorem***:

  *Post-fixed points of an isotone function $\phi$ on a complete lattice form a complete lattice.*

  *Consequently, there is the greatest post-fixed point of $\phi$, which is also the greatest fixed point of $\phi$.*
- ***For every weakly linear system there exists the greatest solution contained in a given fuzzy relation $U$.***

  *If $U$ is a fuzzy quasi-order, then the greatest solution contained in $U$ is also a fuzzy quasi-order.*

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# Computing the greatest solution

**How to effectively compute the greatest solution?** (when the underlying sets are finite)

- ***Kleene Fixed Point Theorem***
- ***Kleene's procedure*** – stabilization of a decreasing sequence $\{\alpha_n\}_{n\in\mathbb{N}}$

  $\alpha_1 = \phi(1), \quad \alpha_{n+1} = \phi(\alpha_n)$

  *if* $\alpha_{n+1} = \alpha_n$, *then* $\alpha_n$ *is the greatest fixed point of* $\phi$
- ***Modified Kleene's procedure***

  $\alpha_1 = \phi(U), \quad \alpha_{n+1} = \alpha_n \wedge \phi(\alpha_n)$

  *if* $\alpha_{n+1} = \alpha_n$, *then* $\alpha_n$ *is the greatest fixed point of* $\phi$ *contained in* $U$

**Termination criteria for the procedure**

- the sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ does not necessary stabilize – *it depends on the structure of truth values*
- ***Termination criteria*** $\equiv$ ***Stabilization criteria*** for the sequence
- ***Sufficient conditions for the stabilization***:
  - ***local finiteness*** of the subalgebra generated by values taken by $R_i$, $R'_i$ and $U$
  - ***Descending Chain Condition*** in this subalgebra
- ***local finiteness*** – all subalgebras of the Gödel structure or a Boolean algebra
- Procedure for computing the greatest *crisp (Boolean) solutions* (which always works)

# Applications in Fuzzy Automata Theory

## Fuzzy finite automata

★ ***Fuzzy finite automaton*** – $\mathscr{A} = (A, X, \{\delta_x^A\}_{x\in X}, \sigma^A, \tau^A)$

- $A$ – finite *set of states*
- $X$ – finite *input alphabet*
- $\{\delta_x^A\}_{x\in X}$ – family of *fuzzy transition relations*

  $\delta_x^A \in L^{A\times A}$ – fuzzy transition relation induced by the input letter $x$
- $\sigma^A \in L^A$ – fuzzy set of *initial states*, $\tau^A \in L^A$ – fuzzy set of *terminal states*

## Behaviour of fuzzy finite automata

★ ***Fuzzy language*** – any fuzzy subset of the free monoid $X^*$

★ ***Fuzzy language recognized by a fuzzy automaton $\mathscr{A}$, behaviour of $\mathscr{A}$*** – $[\![\mathscr{A}]\!] \in L^{X^*}$ given by

$$[\![\mathscr{A}]\!](u) = \sigma^A \circ \delta_u^A \circ \tau^A = \sigma^A \circ \delta_{x_1}^A \circ \cdots \circ \delta_{x_n}^A \circ \tau^A$$

$u = x_1 \dots x_n,\ x_1, \dots, x_n \in X,\qquad \delta_u^A = \delta_{x_1}^A \circ \cdots \circ \delta_{x_n}$ – *composite fuzzy transition relation*

★ Fuzzy automata $\mathscr{A}$ and $\mathscr{B}$ are ***equivalent*** if they have the same behaviour – $[\![\mathscr{A}]\!] = [\![\mathscr{B}]\!]$

# State reduction

## The state reduction problem

- ***The state minimization problem***: *Find an automaton with a minimal number od states equivalent to the given automaton*
- The state minimization problem for fuzzy automata (as well as for nondeterministic automata) is ***computationally hard***
- ***The state reduction problem***: *Construct a "reasonably" small automaton equivalent to the given automaton – it does not have to be minimal, but the construction procedure must be efficient*

## Afterset automata

- $Q$ – fuzzy quasi-order on the set of states $A$

  $aQ \in L^A,\ aQ(b) = Q(a,b)$ – ***afterset*** of $a$ w.r.t. $Q$, or $Q$-afterset

  $A/Q$ – the set of all $Q$-aftersets

- ***afterset fuzzy automaton*** of $\mathscr{A}$ w.r.t. $Q$ – $\mathscr{A}/Q = (A/Q, X, \{\delta_x^{A/Q}\}_{x\in X}, \sigma^{A/Q}, \tau^{A/Q})$

  $\delta_x^{A/Q}(aQ, bQ) = (Q \circ \delta_x^A \circ Q)(a,b)$

  $\sigma^{A/Q}(aQ) = (\sigma^A \circ Q)(a)$

  $\tau^{A/Q}(aQ) = (Q \circ \tau^A)(a)$

# Equivalence of the afterset automaton and the original one

## The general system

- The afterset automaton $\mathscr{A}/Q$ is not necessarily equivalent to $\mathscr{A}$
- *$\mathscr{A}/Q$ is equivalent to $\mathscr{A}$ if and only if $Q$ is a solution of the system*

$$\sigma^A \circ \tau^A = \sigma^A \circ \alpha \circ \tau^A$$

$$\sigma^A \circ \delta^A_{x_1} \circ \cdots \circ \delta^A_{x_n} \circ \tau^A = \sigma^A \circ \alpha \circ \delta^A_{x_1} \circ \alpha \circ \cdots \circ \alpha \circ \delta^A_{x_n} \circ \alpha \circ \tau^A$$

  for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$, where $\alpha$ is an unknown taking values in $L^{A\times A}$
- this system is called the ***general system***
- the general system can have infinitely many equations
- the general system does not necessarily have the greatest solution

## Instances of the general system

- *bigger solutions determine smaller afterset fuzzy automata*
- ***Problem***: *How to find instances having the greatest solutions that can be effectively computed and which are as large as possible*
- ***right invariant fuzzy quasi-orders*** – solutions of the system of weakly linear inequalities $\alpha \circ \delta^A_x \leqslant \delta^A_x \circ \alpha$, $x \in X$, contained in $\tau^A/\tau^A$
- ***left invariant fuzzy quasi-orders*** – solutions of the system of weakly linear inequalities $\delta^A_x \circ \alpha \leqslant \alpha \circ \delta^A_x$, $x \in X$, contained in $\sigma^A\backslash\sigma^A$

## Simulations and bisimulations

- ***Bisimulations*** are the most famous tool used in *computer science*, *mathematical logic* and *set theory* for *modelling equivalence between various systems or processes*
- for fuzzy automata we have defined them through *heterogeneous weakly linear systems* with additional inequalities
- $\mathscr{A} = (A, X, \{\delta_x^A\}_{x\in X}, \sigma^A, \tau^A)$, $\mathscr{B} = (B, X, \{\delta_x^B\}_{x\in X}, \sigma^B, \tau^B)$ – two fuzzy automata
- ***forward simulation*** from $\mathscr{A}$ to $\mathscr{B}$ – solution of the system

$$\alpha^{-1} \circ \delta_x^A \leqslant \delta_x^B \circ \alpha^{-1},\ x \in X, \qquad \alpha^{-1} \circ \tau^A \leqslant \tau^B, \qquad \sigma^A \leqslant \sigma^B \circ \alpha^{-1}$$

- ***backward simulation*** from $\mathscr{A}$ to $\mathscr{B}$ – solution of the system

$$\delta_x^A \circ \alpha \leqslant \alpha \circ \delta_x^B,\ x \in X, \qquad \sigma^A \circ \alpha \leqslant \sigma^B \qquad \tau^A \leqslant \alpha \circ \tau^B,$$

- ***forward bisimulation*** – $R \in L^{A\times B}$ such that *both $R$ and $R^{-1}$ are forward simulations*
- ***backward bisimulation*** – *both $R$ and $R^{-1}$ are backward simulations*
- ***forward-backward bisimulation*** – *$R$ is a forward simulation and $R^{-1}$ is a backward simulation*
- ***backward-forward bisimulation*** – *$R$ is a backward simulation and $R^{-1}$ is a forward simulation*
- Algorithms for testing the existence of simulations and bisimulations between two fuzzy automata, and for computing the greatest ones, when they exist

## Social networks

- ***Fuzzy social network*** – one-mode fuzzy relational system $(A, \{R_i\}_{i\in I})$

  $A$ – set of ***actors***, $\{R_i\}_{i\in I}$ – fuzzy relations between actors

- ***Positional analysis*** – *to identify patterns of relationships that reflect the* ***position*** *or the* ***role*** *of an actor in the network*

  Example: identify positions in a terrorist network on the basis of telephone calls (without insight into the content of the conversation)

- ***Regular equivalences*** – the main tool used in the positional analysis
- Regular equivalences – solutions of ***systems of weakly linear equations***
- our methodology – *efficient procedures for computing the greatest regular equivalences*

# Two-mode systems of FREIs

## Two-mode systems

- $\mathscr{T} = (A, B, \mathscr{R})$, $\mathscr{R} = \{R_i\}_{i\in I} \subset L^{A\times B}$ – two-mode fuzzy relational system
- ***Two-mode systems of FREIs*** are

  $\alpha \circ R_i \leqslant R_i \circ \beta, \quad i \in I$

  $\alpha \circ R_i \geqslant R_i \circ \beta, \quad i \in I$

  $\alpha \circ R_i = R_i \circ \beta, \quad i \in I$

  $\alpha$ and $\beta$ – unknowns taking values in $L^{A\times A}$ and $L^{B\times B}$
- solutions – *pairs of fuzzy relations* – ordered coordinatewise
- equivalent form of $\alpha \circ R_i = R_i \circ \beta, \ i \in I$

  $$(\alpha, \beta) \leqslant \Big( \bigwedge_{i\in I} (R_i \circ \beta)/R_i, \bigwedge_{i\in I} R_i \backslash (\alpha \circ R_i) \Big)$$
- again we use Knaster-Tarski Theorem and the modified Kleene's procedure
- *systems of inequalities* – can be solved directly, using only residuals
- **Applications of two-mode systems**:

  ***two-mode fuzzy social networks*** (e.g., *actor-event*) – ***positional analysis***

  *computing the greatest pairs of regular fuzzy equivalences*

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# Multi-mode systems of FREIs

## Multi-mode systems

- $\mathscr{M} = (A_1, \dots, A_n, \mathscr{R})$, $\mathscr{R} = \{R_i^{j,k} \mid (j,k) \in J,\, i \in I_{j,k}\}$ – multi-mode fuzzy relational system
- ***Multi-mode systems of FREIs*** are

  $$\alpha_j \circ R_i^{j,k} \leqslant R_i^{j,k} \circ \alpha_k, \quad (j,k) \in J, i \in I_{j,k}$$

  $$\alpha_j \circ R_i^{j,k} \geqslant R_i^{j,k} \circ \alpha_k, \quad (j,k) \in J, i \in I_{j,k}$$

  $$\alpha_j \circ R_i^{j,k} = R_i^{j,k} \circ \alpha_k, \quad (j,k) \in J, i \in I_{j,k}$$

  $\alpha_1, \dots, \alpha_n$ – unknowns taking values in $L^{A_1 \times A_1}, \dots, L^{A_n \times A_n}$
- solutions – *n-tuples of fuzzy relations* – ordered coordinatewise
- here we also apply Knaster-Tarski Theorem and the modified Kleene's procedure
- **Applications**:

  ***multi-mode fuzzy social networks – positional analysis***

  *computing the greatest n-tuples of regular fuzzy equivalences*

# Multi-mode systems of FREIs

## Multi-mode systems

- $\mathscr{M} = (A_1,\dots,A_n,\mathscr{R})$, $\mathscr{R} = \{R_i^{j,k} \mid (j,k)\in J,\ i\in I_{j,k}\}$ – multi-mode fuzzy relational system
- ***Multi-mode systems of FREIs*** are

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- solutions – *n-tuples of fuzzy relations* – ordered coordinatewise
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- **Applications**:

  ***multi-mode fuzzy social networks – positional analysis***

  *computing the greatest n-tuples of regular fuzzy equivalences*

# Eigen fuzzy set systems

**Eigen fuzzy set inequalities and equations**

- $\mathscr{O} = (A, \mathscr{R})$, $\mathscr{R} = \{R_i\}_{i\in I}$ – one-mode fuzzy relational system
- $u$ – unknown taking values in $L^A$
- ***Eigen fuzzy set systems*** of inequalities

  $u \circ R_i \leqslant u, \quad i \in I$

  $R_i \circ u \leqslant u, \quad i \in I$

  $u \circ R_i \leqslant u, \ R_i \circ u \leqslant u, \quad i \in I$
- solutions form both opening and closure systems in $L^A$
- the greatest solutions contained in $g \in L^A$ are $g/Q$, $Q\backslash g$ and $g/E = E\backslash g$
- the least solutions containing $g \in L^A$ are $g \circ Q$, $Q \circ g$ and $g \circ E = E \circ g$
- $Q$ – transitive-reflexive closure of $\{R_i\}_{i\in I}$, $E$ – fuzzy equivalence closure of $\{R_i\}_{i\in I}$
- ***Eigen fuzzy set systems*** of equations

  $u \circ R_i = u, \quad i \in I$

  $R_i \circ u = u, \quad i \in I$

  $u \circ R_i = u, \ R_i \circ u = u, \quad i \in I$
- solutions form opening systems in $L^A$
- the greatest solutions – the greatest post-fixed point – Kleene's procedure
- ***Applications***: fuzzy control, image processing, subsystems of fuzzy transition systems

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