Systems of two-sided linear fuzzy relation equations and inequalities and their applications

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Fuzzy relation equations and inequalities (FREIs)

Origins of FRE

★ E. Sanchez (1974) – the first who dealt with FREI’s and systems of FREI’s;
★ he used them in medical research;
★ they were first studied over the Gödel structure
   later, more general structures of truth values were used, including complete residuated lattices

Applications

★ fuzzy control;
★ discrete dynamic systems;
★ knowledge engineering;
★ decision-making;
★ fuzzy information retrieval;
★ fuzzy pattern recognition;
★ image compression and reconstruction;
★ fuzzy automata theory;
★ fuzzy social network analysis, etc.
Fuzzy sets and fuzzy relations

- structure of truth values – complete residuated lattice \( \mathbb{L} = (L, \land, \lor, \otimes, \rightarrow, 0, 1) \)

  residuation property for \( \otimes \) and \( \rightarrow \): \( x \otimes y \leq z \iff x \leq y \rightarrow z \)

- fuzzy subsets of \( A \) – functions of \( A \) to \( L \), \( \mathcal{L}^A \) – all fuzzy subsets of \( A \)

- fuzzy relations between \( A \) and \( B \) – functions of \( A \times B \) to \( L \)

  \( \mathcal{L}^{A \times B} \) – all fuzzy relations between \( A \) and \( B \), \( \mathcal{L}^{A \times A} \) – fuzzy relations on \( A \)

- inclusion: \( f \leq g \iff f(x) \leq g(x) \), for every \( x \in A \)

- intersection and union

\[
\left( \bigwedge_{i \in I} f_i \right)(x) = \bigwedge_{i \in I} f_i(x), \quad \left( \bigvee_{i \in I} f_i \right)(x) = \bigvee_{i \in I} f_i(x)
\]

- for \( R \in \mathcal{L}^{A \times B} \), the inverse fuzzy relation \( R^{-1} \in \mathcal{L}^{B \times A} \) is given by \( R^{-1}(b, a) = R(a, b) \)

- crisp subsets – take only values 0 and 1, \( 2^A \) – all crisp subsets of \( A \)

- if \( A \) and \( B \) are finite with \( |A| = m \) and \( |B| = n \):

  fuzzy subsets of \( A \) \( \equiv \) fuzzy vectors of length \( m \) with entries in \( \mathbb{L} \)

  fuzzy relations between \( A \) and \( B \) \( \equiv \) fuzzy matrices of type \( m \times n \) with entries in \( \mathbb{L} \)
Compositions of fuzzy relations and fuzzy sets

★ **Fuzzy relational composition** – $R \in L^{A \times B}, S \in L^{B \times C}, R \circ S \in L^{A \times C}$

$$(R \circ S)(a, c) = \bigvee_{b \in B} R(a, b) \otimes S(b, c)$$

*matrix product*

★ **Set-relation compositions** – $f \in L^{A}, R \in L^{A \times B}, g \in L^{B}, f \circ R \in L^{B}, R \circ g \in L^{A}$

$$(f \circ R)(b) = \bigvee_{a \in A} f(a) \otimes R(a, b), \quad (R \circ g)(a) = \bigvee_{b \in B} R(a, b) \otimes g(b)$$

*vector-matrix products*

★ **Set-set composition** – $f, g \in L^{A}, f \circ g \in L$ (scalar)

$$f \circ g = \bigvee_{a \in A} f(a) \otimes g(a)$$

*scalar product or dot product*
Residuals of fuzzy relations \( R \in L^{A \times B}, S \in L^{B \times C} \) and \( T \in L^{A \times C} \)

- **Right residual of \( T \) by \( R \) - \( R \setminus T \in L^{B \times C} \)**
  \[
  (R \setminus T)(b, c) = \bigwedge_{a \in A} R(a, b) \rightarrow T(a, c)
  \]

- **Left residual of \( T \) by \( S \) - \( T / S \in L^{A \times B} \)**
  \[
  (T / S)(a, b) = \bigwedge_{c \in C} S(b, c) \rightarrow T(a, c)
  \]

- **Residuation property -**
  \[ R \circ S \leq T \iff S \leq R \setminus T \iff R \leq T / S \]

Residuals of fuzzy sets \( f \in L^{A} \) and \( g \in L^{B} \)

- **Right residual of \( g \) by \( f \) - \( f \setminus g \in L^{A \times B} \)**
  \[ (f \setminus g)(a, b) = f(a) \rightarrow g(b) \]

- **Left residual of \( g \) by \( f \) - \( g / f \in L^{B \times A} \)**
  \[
  (g / f)(b, a) = f(a) \rightarrow g(b) \quad [g / f = (f \setminus g)^{-1}]\]

- **Residuation property -**
  \[ f \circ R \leq g \iff R \leq f \setminus g, \quad S \circ f \leq g \iff S \leq g / f \]

Other types of residuals

Fuzzy relation by fuzzy set, fuzzy set by fuzzy relation, residuals including scalars, etc.
Special types of fuzzy relations

Fuzzy quasi-orders and fuzzy equivalences

★ $R \in L^{A \times A}$ is

- **reflexive** if $R(a, a) = 1$, for all $a \in A$
- **symmetric** if $R(a, b) = R(b, a)$, for all $a, b \in A$
- **transitive** if $R(a, b) \otimes R(b, c) \leq R(a, c)$, for all $a, b, c \in A$

★ **fuzzy quasi-order** – reflexive and transitive fuzzy relation

★ **fuzzy equivalence** – reflexive, symmetric and transitive fuzzy relation
### One-mode fuzzy relational system – $\mathcal{O} = (A, \mathcal{R})$
- $A$ – non-empty set
- $\mathcal{R} = \{R_i\}_{i \in I} \subset L^{A \times A}$

### Two-mode fuzzy relational systems – $\mathcal{T} = (A, B, \mathcal{R})$
- $A, B$ – two different non-empty sets
- $\mathcal{R} = \{R_i\}_{i \in I} \subset L^{A \times B}$

### Multi-mode fuzzy relational systems – $\mathcal{M} = (A_1, \ldots, A_n, \mathcal{R})$
- $A_1, \ldots, A_n$ – non-empty sets
- $\mathcal{R}$ – system of fuzzy relations between $A_j$ and $A_k$ defined for some pairs $(j, k)$
- formally:
  - $J \subseteq [1, n] \times [1, n]$ such that $(\forall j \in [1, n])(\exists k \in [1, n]) (j, k) \in J$ or $(k, j) \in J$
  - $\{I_{j,k}\}_{(j,k) \in J}$ – collection of non-empty sets
  - $\mathcal{R} = \{R_i^{j,k} \mid (j, k) \in J, i \in I_{j,k}\}$
  - $R_i^{j,k} \in L^{A_j \times A_k}$, for all $(j, k) \in J$ and $i \in I_{j,k}$
- complex synthesis of one-mode and two-mode fuzzy relational systems
Systems of linear FREIs

Linear FREIs

- **Sanchez (1974) – linear FREIs**
- one side – linear function of an unknown, another side – a constant (fixed fuzzy relation)
- general form: \( R \circ \alpha \circ S = T, R \circ \alpha \circ S \leq T \) or \( R \circ \alpha \circ S \geq T \), where
  \( R \in L^{A \times B}, S \in L^{C \times D}, T \in L^{A \times D} \)
  \( \alpha \) – unknown taking values in \( L^{B \times C} \)
- some \( R, S, T \) and \( \alpha \) also may be fuzzy sets
- mostly studied: \( R \circ \alpha = S \), where
  \( \alpha \) is an unknown fuzzy relation and \( S \) is a given fuzzy relation, or
  \( \alpha \) is an unknown fuzzy set and \( S \) is a given fuzzy set

Solvability of systems of linear FREIs

  \[ \bigwedge_{i \in I} R_i \setminus T_i / S_i \text{ is the greatest solution of the system } R_i \circ \alpha \circ S_i \leq T_i, i \in I \]
  \[ \text{if } \bigwedge_{i \in I} R_i \setminus T_i / S_i \text{ is a solution of the system } R_i \circ \alpha \circ S_i = T_i, i \in I, \text{ then it is its greatest solution} \]
  otherwise, the system \( R_i \circ \alpha \circ S_i = T_i, i \in I \), does not have any solution
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Solvability of systems of linear FREIs


\[
\bigwedge_{i \in I} R_i \setminus T_i \setminus S_i \text{ is the greatest solution of the system } R_i \circ \alpha \circ S_i \leq T_i, i \in I
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if \( \bigwedge_{i \in I} R_i \setminus T_i \setminus S_i \) is a solution of the system \( R_i \circ \alpha \circ S_i = T_i, i \in I \), then it is its greatest solution

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★ one side – *linear function of an unknown*, another side – a *constant* (fixed fuzzy relation)

★ general form: \( R \circ \alpha \circ S \leq T \) or \( R \circ \alpha \circ S \geq T \), where

\[
R \in L^{A \times B}, \ S \in L^{C \times D}, \ T \in L^{A \times D}
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otherwise, the system \( R_i \circ \alpha \circ S_i = T_i, \ i \in I \), does not have any solution
Systems of weakly linear FREIs

Two-sided linear FREIs

- both sides are linear functions of one unknown or two different unknowns

Homogeneous systems of weakly linear FREIs

- $\mathcal{O} = (A, \{R_i\}_{i \in I})$ – one-mode fuzzy relational system
- Homogeneous systems of weakly linear FREIs are
  \[
  \alpha \circ R_i \leq R_i \circ \alpha, \quad i \in I
  \]
  \[
  \alpha \circ R_i \geq R_i \circ \alpha, \quad i \in I
  \]
  \[
  \alpha \circ R_i = R_i \circ \alpha, \quad i \in I
  \]
  $\alpha$ is an unknown fuzzy relation on the set $A$

Heterogeneous systems of weakly linear FREIs

- $\mathcal{O} = (A, \{R_i\}_{i \in I}), \mathcal{O'} = (A', \{R'_i\}_{i \in I})$ – two one-mode fuzzy relational systems
- Heterogeneous systems of weakly linear FREIs are
  \[
  \alpha \circ R'_i \leq R_i \circ \alpha, \quad i \in I
  \]
  \[
  \alpha \circ R'_i \geq R_i \circ \alpha, \quad i \in I
  \]
  \[
  \alpha \circ R'_i = R_i \circ \alpha, \quad i \in I
  \]
  $\alpha$ is an unknown fuzzy relation between $A$ and $A'$
Systems of weakly linear FREIs

Two-sided linear FREIs

★ both sides are *linear functions of one unknown or two different unknowns*

Homogeneous systems of weakly linear FREIs

★ $\mathcal{O} = (A, \{R_i\}_{i \in I})$ – one-mode fuzzy relational system

★ Homogeneous systems of weakly linear FREIs are

\[
\begin{align*}
\alpha \circ R_i & \leq R_i \circ \alpha, \quad i \in I \\
\alpha \circ R_i & \geq R_i \circ \alpha, \quad i \in I \\
\alpha \circ R_i & = R_i \circ \alpha, \quad i \in I
\end{align*}
\]

$\alpha$ is an unknown fuzzy relation on the set $A$

Homogeneous systems of weakly linear FREIs

★ $\mathcal{O} = (A, \{R_i\}_{i \in I}), \mathcal{O}' = (A', \{R'_i\}_{i \in I})$ – two one-mode fuzzy relational systems

★ Heterogeneous systems of weakly linear FREIs are

\[
\begin{align*}
\alpha \circ R_i' & \leq R_i \circ \alpha, \quad i \in I \\
\alpha \circ R_i' & \geq R_i \circ \alpha, \quad i \in I \\
\alpha \circ R_i' & = R_i \circ \alpha, \quad i \in I
\end{align*}
\]

$\alpha$ is an unknown fuzzy relation between $A$ and $A'$
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Two-sided linear FREIs

★ both sides are \textit{linear functions of one unknown or two different unknowns}

Homogeneous systems of weakly linear FREIs

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Homogeneous systems of weakly linear FREIs

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\]

\( \alpha \) is an unknown fuzzy relation \textit{between} \( A \) and \( A' \)
Greatest solutions of weakly linear systems

Equivalent forms of weakly linear systems

★ using residuals we obtain equivalent forms of weakly linear systems

\[
\alpha \leq \bigwedge_{i \in I} (R_i \circ \alpha)/R_i \\
\alpha \leq \bigvee_{i \in I} (R_i \circ \alpha)/R_i' \\
\alpha \leq \bigwedge_{i \in I} R_i \setminus (\alpha \circ R_i) \\
\alpha \leq \bigvee_{i \in I} R_i \setminus (\alpha \circ R_i') \\
\alpha \leq \bigwedge_{i \in I} (R_i \circ \alpha)/R_i \wedge R_i \setminus (\alpha \circ R_i) \\
\alpha \leq \bigvee_{i \in I} (R_i \circ \alpha)/R_i' \wedge R_i \setminus (\alpha \circ R_i')
\]

Greatest solutions of weakly linear systems

★ expressions on the right-hand sides of these inequalities define isotone functions on \( \alpha \)
★ solutions of the inequalities are post-fixed points of these functions – \( \alpha \leq \phi(\alpha) \)
★ \textbf{Knaster-Tarski Fixed Point Theorem:}

Post-fixed points of an isotone function \( \phi \) on a complete lattice form a complete lattice.
Consequently, there is the greatest post-fixed point of \( \phi \), which is also the greatest fixed point of \( \phi \).
★ \textbf{For every weakly linear system there exists the greatest solution contained in a given fuzzy relation \( U \).}

If \( U \) is a fuzzy quasi-order, then the greatest solution contained in \( U \) is also a fuzzy quasi-order.
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\]

\[
\alpha \leq \bigwedge_{i \in I} (R_i \circ \alpha) / R_i \wedge R_i \setminus (\alpha \circ R_i) \quad \alpha \leq \bigwedge_{i \in I} (R_i \circ \alpha) / R_i' \wedge R_i \setminus (\alpha \circ R_i')
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If \( U \) is a fuzzy quasi-order, then the greatest solution contained in \( U \) is also a fuzzy quasi-order.
How to effectively compute the greatest solution? (when the underlying sets are finite)

**Kleene Fixed Point Theorem**

**Kleene’s procedure** – stabilization of a decreasing sequence \( \{\alpha_n\}_{n \in \mathbb{N}} \)

\[
\alpha_1 = \phi(1), \quad \alpha_{n+1} = \phi(\alpha_n)
\]

if \( \alpha_{n+1} = \alpha_n \), then \( \alpha_n \) is the greatest fixed point of \( \phi \)

**Modified Kleene’s procedure**

\[
\alpha_1 = \phi(U), \quad \alpha_{n+1} = \alpha_n \land \phi(\alpha_n)
\]

if \( \alpha_{n+1} = \alpha_n \), then \( \alpha_n \) is the greatest fixed point of \( \phi \) contained in \( U \)

Termination criteria for the procedure

**the sequence \( \{\alpha_n\}_{n \in \mathbb{N}} \) does not necessary stabilize – it depends on the structure of truth values**

**Termination criteria \( \equiv \) Stabilization criteria** for the sequence

**Sufficient conditions for the stabilization:**

– **local finiteness** of the subalgebra generated by values taken by \( R_i, R'_i \) and \( U \)

– **Descending Chain Condition** in this subalgebra

**local finiteness** – all subalgebras of the Gödel structure or a Boolean algebra

**Procedure for computing the greatest crisp (Boolean) solutions** (which always works)
Fuzzy finite automata

- **Fuzzy finite automaton** \( \mathcal{A} = (A, X, \{ \delta^A_x \}_{x \in X}, \sigma^A, \tau^A) \)
  - \( A \) – finite set of states
  - \( X \) – finite input alphabet
  - \( \{ \delta^A_x \}_{x \in X} \) – family of fuzzy transition relations
    - \( \delta^A_x \in L^A \times A \) – fuzzy transition relation induced by the input letter \( x \)
  - \( \sigma^A \in L^A \) – fuzzy set of initial states
  - \( \tau^A \in L^A \) – fuzzy set of terminal states

Behaviour of fuzzy finite automata

- **Fuzzy language** – any fuzzy subset of the free monoid \( X^* \)
- **Fuzzy language recognized by a fuzzy automaton** \( \mathcal{A} \), behaviour of \( \mathcal{A} \) – \( \llbracket \mathcal{A} \rrbracket \in L^X^* \) given by
  \[
  \llbracket \mathcal{A} \rrbracket(u) = \sigma^A \circ \delta^A_u \circ \tau^A = \sigma^A \circ \delta^A_{x_1} \circ \cdots \circ \delta^A_{x_n} \circ \tau^A
  \]
  - \( u = x_1 \ldots x_n, x_1, \ldots, x_n \in X \)
  - \( \delta^A_u = \delta^A_{x_1} \circ \cdots \circ \delta^A_{x_n} \) – composite fuzzy transition relation
- **Fuzzy automata** \( \mathcal{A} \) and \( \mathcal{B} \) are equivalent if they have the same behaviour – \( \llbracket \mathcal{A} \rrbracket = \llbracket \mathcal{B} \rrbracket \)
State reduction

The state reduction problem

★ **The state minimization problem:** Find an automaton with a minimal number of states equivalent to the given automaton

★ The state minimization problem for fuzzy automata (as well as for nondeterministic automata) is *computationally hard*

★ **The state reduction problem:** Construct a "reasonably" small automaton equivalent to the given automaton – it does not have to be minimal, but the construction procedure must be efficient

Afterset automata

★ $Q$ – fuzzy quasi-order on the set of states $A$

$$aQ \in L^A, \ aQ(b) = Q(a, b) – \text{afterset } \text{of } a \text{ w.r.t. } Q, \text{ or } Q\text{-afterset}$$

$A/Q$ – the set of all $Q$-aftersets

★ *afterset fuzzy automaton* of $A$ w.r.t. $Q$ – $A/Q = (A/Q, X, \{\delta^A/Q\}_{x \in X}, \sigma^A/Q, \tau^A/Q)$

$$\delta^A/Q(aQ, bQ) = (Q \circ \delta^A_x \circ Q)(a, b)$$

$$\sigma^A/Q(aQ) = (\sigma^A \circ Q)(a)$$

$$\tau^A/Q(aQ) = (Q \circ \tau^A)(a)$$
Equivalence of the afterset automaton and the original one

The general system

- The afterset automaton $\mathcal{A}/Q$ is not necessarily equivalent to $\mathcal{A}$
- $\mathcal{A}/Q$ is equivalent to $\mathcal{A}$ if and only if $Q$ is a solution of the system

$$
\sigma^A \circ \tau^A = \sigma^A \circ \alpha \circ \tau^A
$$

$$
\sigma^A \circ \delta^A_{x_1} \circ \cdots \circ \delta^A_{x_n} \circ \tau^A = \sigma^A \circ \alpha \circ \delta^A_{x_1} \circ \alpha \circ \cdots \circ \alpha \circ \delta^A_{x_n} \circ \alpha \circ \tau^A
$$

for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$, where $\alpha$ is an unknown taking values in $L^{A \times A}$

- this system is called the general system

- the general system can have infinitely many equations

- the general system does not necessarily have the greatest solution

Instances of the general system

- bigger solutions determine smaller afterset fuzzy automata

- **Problem**: How to find instances having the greatest solutions that can be effectively computed and which are as large as possible

- **right invariant fuzzy quasi-orders** – solutions of the system of weakly linear inequalities $\alpha \circ \delta^A_x \leq \delta^A_x \circ \alpha$, $x \in X$, contained in $\tau^A/\tau^A$

- **left invariant fuzzy quasi-orders** – solutions of the system of weakly linear inequalities $\delta^A_x \circ \alpha \leq \alpha \circ \delta^A_x$, $x \in X$, contained in $\sigma^A/\sigma^A$
Equivalence, simulation, bisimulation

Simulations and bisimulations

- **Bisimulations** are the most famous tool used in computer science, mathematical logic and set theory for modelling equivalence between various systems or processes.
- For fuzzy automata, we have defined them through heterogeneous weakly linear systems with additional inequalities.
- \( \mathcal{A} = (A, X, \{\delta^A_x\}_{x \in X}, \sigma^A, \tau^A) \), \( \mathcal{B} = (B, X, \{\delta^B_x\}_{x \in X}, \sigma^B, \tau^B) \) – two fuzzy automata.
- **Forward simulation** from \( \mathcal{A} \) to \( \mathcal{B} \) – solution of the system:
  \[
  \alpha^{-1} \circ \delta^A_x \leq \delta^B_x \circ \alpha^{-1}, \quad x \in X, \quad \alpha^{-1} \circ \tau^A \leq \tau^B, \quad \sigma^A \leq \sigma^B \circ \alpha^{-1}.
  \]
- **Backward simulation** from \( \mathcal{A} \) to \( \mathcal{B} \) – solution of the system:
  \[
  \delta^A_x \circ \alpha \leq \alpha \circ \delta^B_x, \quad x \in X, \quad \sigma^A \circ \alpha \leq \sigma^B \tau^A \leq \alpha \circ \tau^B.
  \]
- **Forward bisimulation** – \( R \in L^{A \times B} \) such that both \( R \) and \( R^{-1} \) are forward simulations.
- **Backward bisimulation** – both \( R \) and \( R^{-1} \) are backward simulations.
- **Forward-backward bisimulation** – \( R \) is a forward simulation and \( R^{-1} \) is a backward simulation.
- **Backward-forward bisimulation** – \( R \) is a backward simulation and \( R^{-1} \) is a forward simulation.
- Algorithms for testing the existence of simulations and bisimulations between two fuzzy automata, and for computing the greatest ones, when they exist.
Social networks

- **Fuzzy social network** – one-mode fuzzy relational system \((A, \{R_i\}_{i \in I})\)
  
  - \(A\) – set of actors, \(\{R_i\}_{i \in I}\) – fuzzy relations between actors

- **Positional analysis** – to identify patterns of relationships that reflect the **position** or the **role** of an actor in the network

  Example: identify positions in a terrorist network on the basis of telephone calls (without insight into the content of the conversation)

- **Regular equivalences** – the main tool used in the positional analysis
  
  - Regular equivalences – solutions of systems of **weakly linear equations**
  
  - our methodology – **efficient procedures for computing the greatest regular equivalences**
Two-mode systems of FREIs

Two-mode systems

- \( \mathcal{T} = (A, B, \mathcal{R}), \mathcal{R} = \{R_i\}_{i \in I} \subset L^{A \times B} \) – two-mode fuzzy relational system
- Two-mode systems of FREIs are
  \[ \alpha \circ R_i \leq R_i \circ \beta, \quad i \in I \]
  \[ \alpha \circ R_i \geq R_i \circ \beta, \quad i \in I \]
  \[ \alpha \circ R_i = R_i \circ \beta, \quad i \in I \]
- \( \alpha \) and \( \beta \) – unknowns taking values in \( L^{A \times A} \) and \( L^{B \times B} \)
- solutions – pairs of fuzzy relations – ordered coordinatewise
- equivalent form of \( \alpha \circ R_i = R_i \circ \beta, \quad i \in I \)
  \[ (\alpha, \beta) \leq \left( \bigwedge_{i \in I} (R_i \circ \beta)/R_i, \bigwedge_{i \in I} R_i \setminus (\alpha \circ R_i) \right) \]
- again we use Knaster-Tarski Theorem and the modified Kleene’s procedure
- systems of inequalities – can be solved directly, using only residuals
- Applications of two-mode systems:
  two-mode fuzzy social networks (e.g., actor-event) – positional analysis
  computing the greatest pairs of regular fuzzy equivalences
Two-mode systems of FREIs

Two-mode systems

- \( \mathcal{T} = (A, B, \mathcal{R}), \mathcal{R} = \{R_i\}_{i \in I} \subset L^{A \times B} \) – two-mode fuzzy relational system
- **Two-mode systems of FREIs** are
  - \( \alpha \circ R_i \leq R_i \circ \beta, \ i \in I \)
  - \( \alpha \circ R_i \geq R_i \circ \beta, \ i \in I \)
  - \( \alpha \circ R_i = R_i \circ \beta, \ i \in I \)
  - \( \alpha \) and \( \beta \) – unknowns taking values in \( L^{A \times A} \) and \( L^{B \times B} \)
- solutions – *pairs of fuzzy relations* – ordered coordinatewise
- equivalent form of \( \alpha \circ R_i = R_i \circ \beta, \ i \in I \)
  - \( (\alpha, \beta) \leq \left( \bigwedge_{i \in I} (R_i \circ \beta) \big/ R_i, \bigwedge_{i \in I} R_i \setminus (\alpha \circ R_i) \right) \)
- again we use Knaster-Tarski Theorem and the modified Kleene’s procedure
- **systems of inequalities** – can be solved directly, using only residuals
- **Applications of two-mode systems:**
  - *two-mode fuzzy social networks* (e.g., actor-event) – *positional analysis*
  - computing the greatest pairs of regular fuzzy equivalences
Multi-mode systems

\[ M = (A_1, \ldots, A_n, R), \quad R = \{ R_{i,j,k}^i \mid (j, k) \in J, i \in I_{j,k} \} \] – multi-mode fuzzy relational system

★ **Multi-mode systems of FREIs** are

\[
\begin{align*}
\alpha_j \circ R_{i,j,k}^i & \leq R_{i,j,k}^i \circ \alpha_k, \quad (j, k) \in J, i \in I_{j,k} \\
\alpha_j \circ R_{i,j,k}^i & \geq R_{i,j,k}^i \circ \alpha_k, \quad (j, k) \in J, i \in I_{j,k} \\
\alpha_j \circ R_{i,j,k}^i & = R_{i,j,k}^i \circ \alpha_k, \quad (j, k) \in J, i \in I_{j,k}
\end{align*}
\]

\[ \alpha_1, \ldots, \alpha_n \] – unknowns taking values in \( L^{A_1 \times A_1}, \ldots, L^{A_n \times A_n} \)

★ solutions – \( n \)-tuples of fuzzy relations – ordered coordinatewise

★ here we also apply Knaster-Tarski Theorem and the modified Kleene’s procedure

★ **Applications:**

*multi-mode fuzzy social networks – positional analysis*

*computing the greatest \( n \)-tuples of regular fuzzy equivalences*
Multi-mode systems

★ $\mathcal{M} = (A_1, \ldots, A_n, \mathcal{R})$ is a $\mathcal{R} = \{R^i_{jk} | (j, k) \in J, i \in I_{j,k}\}$ – multi-mode fuzzy relational system

★ **Multi-mode systems of FREIs** are

\[
\begin{align*}
\alpha_j \circ R^i_{jk} &\leq R^i_{jk} \circ \alpha_k, \quad (j, k) \in J, i \in I_{j,k} \\
\alpha_j \circ R^i_{jk} &\geq R^i_{jk} \circ \alpha_k, \quad (j, k) \in J, i \in I_{j,k} \\
\alpha_j \circ R^i_{jk} &= R^i_{jk} \circ \alpha_k, \quad (j, k) \in J, i \in I_{j,k}
\end{align*}
\]

$\alpha_1, \ldots, \alpha_n$ – unknowns taking values in $L^{A_1 \times A_1}, \ldots, L^{A_n \times A_n}$

★ solutions – $n$-tuples of fuzzy relations – ordered coordinatewise

★ here we also apply Knaster-Tarski Theorem and the modified Kleene’s procedure

★ **Applications:**

- *multi-mode fuzzy social networks* – positional analysis
  computing the greatest $n$-tuples of regular fuzzy equivalences
Eigen fuzzy set systems

Eigen fuzzy set inequalities and equations

★ $\mathcal{O} = (A, \mathcal{R})$, $\mathcal{R} = \{R_i\}_{i \in I}$ – one-mode fuzzy relational system
★ $u$ – unknown taking values in $L^A$
★ *Eigen fuzzy set systems* of inequalities
  
  \[
  u \circ R_i \leq u, \quad i \in I
  \]
  
  \[
  R_i \circ u \leq u, \quad i \in I
  \]
  
  \[
  u \circ R_i \leq u, \quad R_i \circ u \leq u, \quad i \in I
  \]
★ solutions form both opening and closure systems in $L^A$
★ the greatest solutions contained in $g \in L^A$ are $g/Q$, $Q\setminus g$ and $g/E = E\setminus g$
★ the least solutions containing $g \in L^A$ are $g \circ Q$, $Q \circ g$ and $g \circ E = E \circ g$
★ $Q$ – transitive-reflexive closure of $\{R_i\}_{i \in I}$, $E$ – fuzzy equivalence closure of $\{R_i\}_{i \in I}$
★ *Eigen fuzzy set systems* of equations
  
  \[
  u \circ R_i = u, \quad i \in I
  \]
  
  \[
  R_i \circ u = u, \quad i \in I
  \]
  
  \[
  u \circ R_i = u, \quad R_i \circ u = u, \quad i \in I
  \]
★ solutions form opening systems in $L^A$
★ the greatest solutions – the greatest post-fixed point – Kleene’s procedure
★ *Applications*: fuzzy control, image processing, subsystems of fuzzy transition systems
References

**Weakly linear systems:**


**Two-mode systems:**


**Multi-mode systems:**


**Related subjects:**

Applications – State reduction:


Applications – Simulation and bisimulation: