Random Walks, Cogrowth and Amenability of Semigroups[1]

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AAA 94 + NSAC 2017
Novi Sad, 16 June 2017

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- *M is a bounded symmetric operator on $\ell_2(S)$, of norm ≤ 1 (hence spectral radius ≤ 1);*
- *M has spectral radius $1 \iff S$ is* **amenable**.

Definition

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Definition (Day 1957)

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- the bicyclic monoid

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Theorem (Grigorchuk 1980, Cohen 1982)

$\kappa \leq 2|X| - 1$, *with equality if and only if G is amenable.*

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So $\lambda_1 = 2$.

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If $s \leq_{\mathcal{J}} t$ then $\lambda_s \geq \lambda_t$. In particular, local cogrowth is a $\mathcal{J}$-class invariant.

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Proposition/Exercise

If S has an element of maximal local cogrowth (i.e. local cogrowth $|X|$), then the set of all such elements forms a minimal ideal of S.

Example

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One can equivalently replace $\gamma(n)$ with

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Lemma

For any $0 \leq \kappa < \gamma$ there exists $C > 0$ such that $\gamma(n) \geq \gamma'(n) > C\kappa^n$ for all even n.

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Example

For the bicyclic monoid, $\gamma = 2$.

Theorem (building on Elder-Rechnitzer-Wong 2012 building on Grigorchuk 1980 / Cohen 1982 building on Kesten 1959)

If S is a group and X is a symmetric generating set then S has maximal global cogrowth if and only if S is amenable.

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Theorem (Gray & K. 2017)

Suppose S is a monoid with maximal global cogrowth with respect to some finite choice of generators. Then for every finite $K \subseteq S$, the monoid S has maximal global cogrowth with respect to some choice of generators (with multiplicity) containing K.

Theorem (Gray & K. 2016)

If S has maximal global cogrowth then the associated Markov operator M on $\ell_2(S)$ has spectral radius ≥ 1.

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As a consequence with have one implication of Kesten's theorem for semigroups:

Corollary

If S is right amenable then the associated Markov operator M on $\ell_2(S)$ has spectral radius ≥ 1.

Theorem (Gray & K. 2015)

Suppose S is right reversible and the max. left cancellative quotient of S has a minimal ideal. If S has maximal global cogrowth then S is right amenable.

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A finitely generated inverse semigroup (or group!) of maximal global cogrowth is (left and right) amenable. (No symmetry assumption on the generating set!)

Corollary

A finitely generated right reversible semigroup of maximal **local** *cogrowth is right amenable.*

Near Right Cancellativity

Definition

$E \subseteq S$ is **right thick** if for all finite $F \subseteq S$ there exists $t \in S$ with $tF \subseteq E$.

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Remark

These semigroups **behave dynamically** *like right cancellative semigroups.*

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Theorem (Gray & K. 2017)

If S is a right reversible, near right cancellative monoid of maximal global cogrowth then S is right amenable.

The Big Picture

right strong Følner condition

maximal local cogrowth

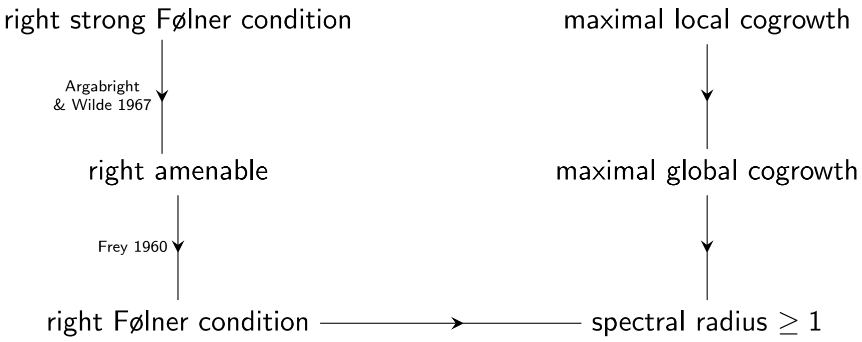
right amenable

maximal global cogrowth

right Følner condition

spectral radius ≥ 1

The Big Picture



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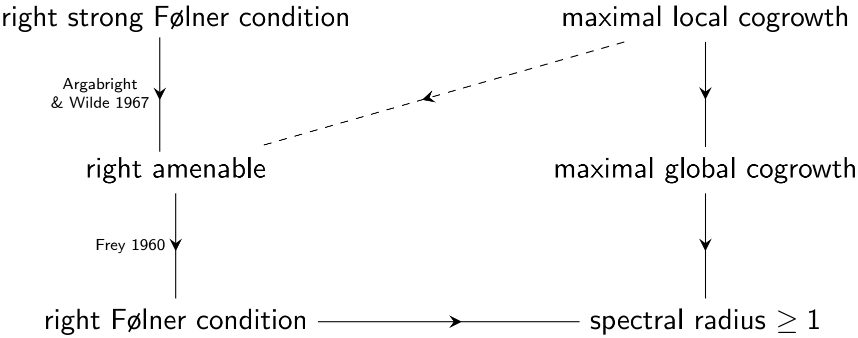

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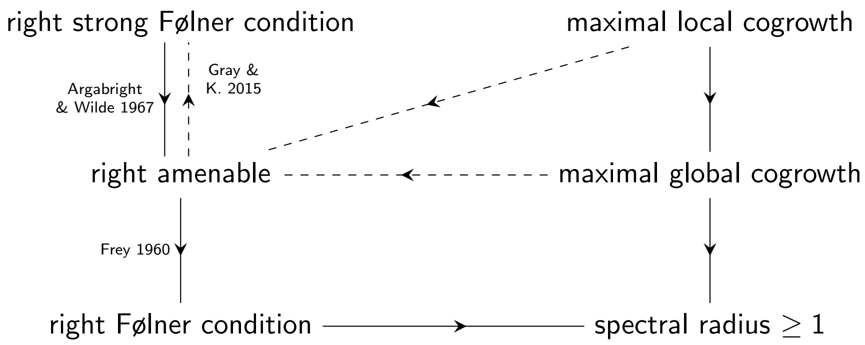

$\longrightarrow$ implications hold for all semigroups.

$\dashrightarrow$ implications hold for right reversible near right cancellative semigroups.

More detail....

- R.D.Gray and M.Kambites, *Amenability and geometry of semigroups*, Trans. AMS (to appear), preprint at `arXiv:1505.06139 [math.GR]`
- R.D.Gray and M.Kambites, *On cogrowth, amenability and the spectral radius of a random walk on a semigroup*, preprint at `arXiv:1706.01313 [math.GR]`

Also relevant....

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- Other ways to define amenability for inverse semigroups.