Fair semigroups

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monoid $\Rightarrow$ local units $\Rightarrow$ weak local units $\Rightarrow$ factorisable

Let S be a semigroup. A **right S-act** is a set A with a mapping (action of S on A)

$$A \times S \to A, \;\; (a, s) \mapsto as$$

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A right S-act A_S is called **unitary** if $AS = A$, that is,

$$(\forall a \in A)(\exists a' \in A)(\exists s \in S)\ a = a's.$$

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Definition

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Proposition

A semigroup S has weak local units if and only if S is fair and factorisable.

Theorem

A semigroup S is right fair if and only if

$$(\forall (s_i)_{i\in\mathbb{N}} \in S^{\mathbb{N}})(\exists n \in \mathbb{N})(\exists u \in S)\ s_n \dots s_2 s_1 = s_n \dots s_2 s_1 u.$$

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Corollary

Free semigroups and free commutative semigroups are not fair.

A semigroup S is called an **epigroup** if

$$(\forall s \in S)(\exists n \in \mathbb{N})\ s^n \text{ belongs to a subgroup of } S.$$

Problem

Which epigroups are (right) fair?

Example

Consider a semigroup $S = \{0, a, e\}$ with the multiplication table

$$\begin{array}{c|ccc} & 0 & a & e \\ \hline 0 & 0 & 0 & 0 \\ a & 0 & 0 & a \\ e & 0 & 0 & e \end{array}.$$

The element e is a right identity of S; in particular, S is factorisable and right fair. On the other hand, S is not left fair because the subproducts of the sequence

$$a, e, e, e, \ldots$$

are all equal to a and $sa = 0$ for all $s \in S$.

Let $U(S_S)$ be the union of all right ideals I of a semigroup S which are right unitary, that is, $IS = I$. Then $U(S_S)$ is the largest right ideal of S which is right unitary. Dually one can consider the left ideal $U({}_S S)$.

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Lemma

If S is a right fair semigroup then $U(S_S)$ is a two-sided ideal of S.

Lemma

Let S be a fair semigroup. For every $s \in S$, the following assertions are equivalent.

1. $s \in U(S_S)$.
2. $s = su$ for some $u \in S$.
3. $s \in U({}_SS)$.
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By this lemma, $U(S_S) = U({}_S S)$, so we denote this set by $U(S)$ and call it **the unitary part** of the fair semigroup S.

Corollary

If S is a fair semigroup then the set

$$U(S) = \{s \in S \mid s = su = vs \text{ for some } u, v \in S\}$$

is a two-sided ideal of S. Moreover, $U(S)$ is a semigroup with weak local units (hence also a fair semigroup).

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6. A direct product of finitely many fair semigroups is a fair semigroup.

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2. A direct product of infinitely many fair semigroups is not necessarily a fair semigroup.
3. An ideal extension of a fair semigroup by a fair semigroup need not be itself fair.
4. A finite semigroup generated by two elements need not be fair: $\{a, e\}$ is a generating set for the non-fair semigroup $S = \{0, a, e\}$ presented earlier.

Lemma

If a semigroup S has the descending chain condition (DCC) for principal left ideals then for each sequence $(s_i)_{i\in\mathbb{N}} \in S^{\mathbb{N}}$ there exist $n \in \mathbb{N}$ and $u \in S^1$ such that

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Corollary

Every finite commutative semigroup is a fair semigroup.

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The category of all firm right S-acts is denoted by FAct_S.

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If S and T are strongly Morita equivalent then they are factorisable.

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Corollary

Finite monogenic semigroup is Morita equivalent to its group part. Hence Morita equivalent semigroups need not be strongly Morita equivalent.

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3. *$U(S_S)$ and $U(T_T)$ are strongly Morita equivalent.*

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Thank you!