A semigroup-theoretical approach to the study of generalized inverses

Miroslav Ćirić

Department of Computer Science
Faculty of Sciences and Mathematics
University of Niš, Serbia
miroslav.ciric@pmf.edu.rs

joint work with

Predrag Stanimirović and Jelena Ignjatović

94. Arbeitstagung Allgemeine Algebra + 5th Novi Sad Algebraic Conference
June 17th, 2017, Novi Sad, Serbia

Origins of generalized inverses

Historical notes

- ⋆ ***E. I. Fredholm*** (1903) – generalized inverses of integral operators;
- ⋆ ***D. Hilbert*** (1904) – differential operators, ***A. Hurwitz*** (1912), and others;
- ⋆ ***E. H. Moore*** (1920 or earlier) – generalized inverses of matrices (*general reciprocal*);
- ⋆ Moore's work has not attracted more attention until the 1950s;
- ⋆ ***A. Bjerhammar*** (1951) – links with solutions of linear systems;
  - – *least-squares solutions (approximate)* and *minimum-norm solutions*;
- ⋆ ***R. Penrose*** (1955) – generalized inverses as solutions of algebraic equations;
  - – ***Moore-Penrose equations***;
  - – in abstract algebraic structures – semigroups, rings, Banach algebras, C^*-algebras, etc.;

Applications

- ⋆ *solving matrix equations*;
- ⋆ *solving singular differential and difference equations*;
- ⋆ *the investigation of Cesaro-Neumann iterations*;
- ⋆ *the least squares approximation*;
- ⋆ *finite Markov chains, cryptography, statistics*, etc.

Moore-Penrose equations

Moore-Penrose equations

- ⋆ S – an involution semigroup;
- ⋆ $a \in S$ – fixed element, x – unknown taking values in S;
- ⋆ ***Moore-Penrose equations***:

  (1) $axa = a$
  (2) $xax = x$;
  (3) $(ax)^* = ax$
  (4) $(xa)^* = xa$

- ⋆ Additionally, we consider the equation

  (5) $ax = xa$

γ-inverses

- ⋆ for $\gamma \subseteq \{1, 2, 3, 4, 5\}$, by $\langle\gamma\rangle$ we denote the system consisting of the equations (i), for all $i \in \gamma$;
- ⋆ if $\langle\gamma\rangle$ has a solution, then a is called ***γ-invertible***.
- ⋆ in this case, solutions of $\langle\gamma\rangle$ are called ***γ-inverses*** of a.

Terminology and notation

Terminology

{1}-inverse	***g-inverse*** (“*generalized inverse*”) or ***inner inverse***
{2}-inverse	***outer inverse***
$\{1,2\}$-inverse	***reflexive g-inverse*** or ***Thierrin-Vagner inverse***
$\{1,3\}$-inverse	***last-squares g-inverse***
$\{1,4\}$-inverse	***minimum-norm g-inverse***
$\{1,2,3,4\}$-inverse	***Moore-Penrose inverse*** or ***MP-inverse***
$\{1,2,5\}$-inverse	***group inverse***

⋆ When exist, the Moore-Penrose inverse and the group inverse of a are ***unique***.

Notation

$a^\dagger$	Moore-Penrose inverse of a
$a^\#$	group inverse of a
$a\gamma$	the set of all γ-inverses of a
$a\gamma_X$	the set of all γ-inverses of a contained in the set X
$S^{(1)}$	the set of all {1}-invertible elements ($Reg(S)$)
$X^\bullet$	the set of all idempotents contained in the set X

Matrices with prescribed range and null-space

Range, null space, rank

- $A \in \mathbb{C}^{m\times n}$ – a complex matrix of type $m \times n$;
- ***range*** or ***image*** of A

$$\boldsymbol{R}(A) = \{\boldsymbol{y} \in \mathbb{C}^m \mid A\boldsymbol{x} = \boldsymbol{y}, \text{ for some } \boldsymbol{x} \in \mathbb{C}^n\}$$

- ***null space*** or ***kernel*** of A

$$\boldsymbol{N}(A) = \{\boldsymbol{x} \in \mathbb{C}^n \mid A\boldsymbol{x} = \boldsymbol{0}\}$$

- rank(A) – the ***rank*** of A (the dimension of the *column space* and the *row space* of A).

Generalized inverses with prescribed range and null space

- $T \subseteq \mathbb{C}^m$, $S \subseteq \mathbb{C}^n$ – given subspaces, $A \in \mathbb{C}^{m\times n}$;
- $A^{(2)}_{T,S}$ – the $\{2\}$-inverse of A with prescribed range T and null space S (if it exists)
- $A^{(1,2)}_{T,S}$ – the $\{1,2\}$-inverse of A with prescribed range T and null space S (if it exists)

$$A^{\dagger} = A^{(2)}_{\boldsymbol{R}(A^*),\boldsymbol{N}(A^*)} = A^{(1,2)}_{\boldsymbol{R}(A^*),\boldsymbol{N}(A^*)} \qquad A^{\#} = A^{(2)}_{\boldsymbol{R}(A),\boldsymbol{N}(A)} = A^{(1,2)}_{\boldsymbol{R}(A),\boldsymbol{N}(A)}$$

- A^* – the ***conjugate transpose*** of A

Ring-theoretical generalizations

Generalized inverses with prescribed idempotents

- ⋆ ***D. Djordjević, Y. Wei***, Communications in Algebra 33 (2005) 3051–3060

  outer inverses with prescribed idempotents ($xax = x$ and idempotents ax, xa are prescribed)

- ⋆ ***B. Načevska, D. Djordjević***, Communications in Algebra 39 (2011) 634–646

  inner inverses with prescribed idempotents ($a = axa$ and idempotents ax, xa are prescribed)

Generalized inverses with prescribed ideals

- ⋆ ***D. Mosić, D. Djordjević, G. Kantún-Montiel***, Electronic Journal in Linear Algebra 27 (2014) 272–283

- ⋆ ***G. Kantún-Montiel***, Linear and Multilinear Algebra 62 (2014) 1187–1196

  outer inverses with prescribed kernel ideals (left and right annihilators)

  outer inverses with prescribed image ideals (principal left and right ideals)

Semigroup-theoretical generalization

Semigroup of matrices $M_\varnothing(\mathbb{C})$

Let $M(\mathbb{C})$ be the set of ***all matrices of any type*** with entries in $\mathbb{C}$, i.e.,

$$M(\mathbb{C}) = \bigcup_{m,n\in\mathbb{N}} \mathbb{C}^{m\times n},$$

Let $M_\varnothing(\mathbb{C}) = M(\mathbb{C}) \cup \{\varnothing\}$, where $\varnothing \notin M(\mathbb{C})$.

For the sake of convenience, we call $\varnothing$ the ***empty matrix***.

The multiplication in $M_\varnothing(\mathbb{C})$ is defined using the standard procedure for converting a partial semigroup into a semigroup:

- ⋆ the product in $M_\varnothing(\mathbb{C})$ coincides with the ***ordinary matrix product***, whenever it is defined;
- ⋆ in all other cases the product is equal to the empty matrix $\varnothing$.

With respect to this multiplication, $M_\varnothing(\mathbb{C})$ is a semigroup with the zero $\varnothing$.

We call $M_\varnothing(\mathbb{C})$ the ***semigroup of matrices*** with entries in $\mathbb{C}$.

Semigroup-theoretical generalization (cont.)

Green's equivalences in the semigroup of matrices

For any $A, B \in M(\mathbb{C})$ *we have*

$$\begin{aligned} A\,\mathscr{R}\,B &\Leftrightarrow \boldsymbol{R}(A) = \boldsymbol{R}(B), \\ A\,\mathscr{L}\,B &\Leftrightarrow \boldsymbol{N}(A) = \boldsymbol{N}(B), \\ A\,\mathscr{H}\,B &\Leftrightarrow \boldsymbol{R}(A) = \boldsymbol{R}(B)\ \&\ \boldsymbol{N}(A) = \boldsymbol{N}(B), \\ A\,\mathscr{D}\,B &\Leftrightarrow \operatorname{rank}(A) = \operatorname{rank}(B) \end{aligned}$$

On the other hand,

$$D_\varnothing = R_\varnothing = L_\varnothing = H_\varnothing = \{\varnothing\}.$$

Our mission

Outer inverses belonging to prescribed Green's equivalence classes

- ***M. Ćirić, P. Stanimirović, J. Ignjatović***, *Outer inverses in semigroups belonging to prescribed Green's equivalence classes*, to appear.

Outer inverses in the prescribed Green's $\mathcal{R}$-class

In the sequel, let S be a semigroup and let $a, b \in S$.

Theorem 1

An element $x \in S$ is an outer inverse of a contained in the $\mathscr{R}$-class R_b if and only if

$$x \in bS \quad \text{and} \quad xab = b.$$

Theorem 2 – *The outer inverse in the prescribed $\mathscr{R}$-class*

The following statements are equivalent:

(i) *there exists an outer inverse of a contained in the $\mathscr{R}$-class R_b;*

(ii) *there exists $u \in S$ such that $b = buab$;*

(iii) $b \in S^{(1)}$ *and* $ab \in L_b$;

(iv) $ab \in S^{(1)}$ *and* $b(ab)^{(1)}ab = b$, *for some (eqivalently every)* $(ab)^{(1)} \in ab\{1\}$.

If these statements are true, then

$$a\{2\}_{R_b} = \{bu \mid u \in S \text{ such that } b = buab\} = \{b(ab)^{(1)} \mid (ab)^{(1)} \in ab\{1\}\}.$$

Outer inverses in the prescribed Green's $\mathcal{R}$-class

In the sequel, let S be a semigroup and let $a, b \in S$.

Theorem 1

An element $x \in S$ is an outer inverse of a contained in the $\mathscr{R}$-class R_b if and only if

$$x \in bS \quad \text{and} \quad xab = b.$$

Theorem 2 – *The outer inverse in the prescribed $\mathscr{R}$-class*

The following statements are equivalent:

(i) *there exists an outer inverse of a contained in the $\mathscr{R}$-class R_b;*

(ii) *there exists $u \in S$ such that $b = buab$;*

(iii) $b \in S^{(1)}$ *and* $ab \in L_b$;

(iv) $ab \in S^{(1)}$ *and* $b(ab)^{(1)}ab = b$, *for some (eqivalently every)* $(ab)^{(1)} \in ab\{1\}$.

If these statements are true, then

$$a\{2\}_{R_b} = \{bu \mid u \in S \text{ such that } b = buab\} = \{b(ab)^{(1)} \mid (ab)^{(1)} \in ab\{1\}\}.$$

Outer inverses in the prescribed Green's $\mathcal{R}$-class

In the sequel, let S be a semigroup and let $a, b \in S$.

Theorem 1

An element $x \in S$ is an outer inverse of a contained in the $\mathscr{R}$-class R_b if and only if

$$x \in bS \quad \text{and} \quad xab = b.$$

Theorem 2 – *The outer inverse in the prescribed $\mathscr{R}$-class*

The following statements are equivalent:

(i) *there exists an outer inverse of a contained in the $\mathscr{R}$-class R_b;*

(ii) *there exists $u \in S$ such that $b = buab$;* ***linear equation***

(iii) $b \in S^{(1)}$ *and* $ab \in L_b$;

(iv) $ab \in S^{(1)}$ *and* $b(ab)^{(1)}ab = b$, *for some (eqivalently every)* $(ab)^{(1)} \in ab\{1\}$.

If these statements are true, then

$$a\{2\}_{R_b} = \{bu \mid u \in S \text{ such that } b = buab\} = \{b(ab)^{(1)} \mid (ab)^{(1)} \in ab\{1\}\}.$$

Outer inverses in the prescribed Green's $\mathcal{R}$-class

In the sequel, let S be a semigroup and let $a, b \in S$.

Theorem 1

An element $x \in S$ is an outer inverse of a contained in the $\mathscr{R}$-class R_b if and only if

$$x \in bS \quad \text{and} \quad xab = b.$$

Theorem 2 – *The outer inverse in the prescribed $\mathscr{R}$-class*

The following statements are equivalent:

(i) *there exists an outer inverse of a contained in the $\mathscr{R}$-class R_b;*

(ii) *there exists $u \in S$ such that $b = buab$;* **linear equation**

(iii) $b \in S^{(1)}$ *and* $ab \in L_b$*;*

(iv) $ab \in S^{(1)}$ *and* $b(ab)^{(1)}ab = b$*, for some (eqivalently every)* $(ab)^{(1)} \in ab\{1\}$.

If these statements are true, then

$$a\{2\}_{R_b} = \{bu \mid u \in S \text{ such that } b = buab\} = \{b(ab)^{(1)} \mid (ab)^{(1)} \in ab\{1\}\}.$$

Outer inverses in the prescribed Green's $\mathcal{R}$-class

In the sequel, let S be a semigroup and let $a, b \in S$.

Theorem 1

An element $x \in S$ is an outer inverse of a contained in the $\mathscr{R}$-class R_b if and only if

$$x \in bS \quad \text{and} \quad xab = b.$$

Theorem 2 – *The outer inverse in the prescribed $\mathscr{R}$-class*

The following statements are equivalent:

(i) *there exists an outer inverse of a contained in the $\mathscr{R}$-class R_b;*

(ii) *there exists $u \in S$ such that $b = buab$;* ***linear equation***

(iii) $b \in S^{(1)}$ *and* $ab \in L_b$;

(iv) $ab \in S^{(1)}$ *and* $b(ab)^{(1)}ab = b$*, for some (eqivalently every)* $(ab)^{(1)} \in ab\{1\}$.

If these statements are true, then

$$a\{2\}_{R_b} = \{bu \mid u \in S \text{ such that } b = buab\} = \{b(ab)^{(1)} \mid (ab)^{(1)} \in ab\{1\}\}.$$

Theorem 3 – *The inner inverse in the prescribed principal right ideal*

The following statements are equivalent:

(i) *there exists an inner inverse of a contained in the principal right ideal $R(b)$;*

(ii) *there exists $u \in S$ such that $a = abua$;*

(iii) $a \in S^{(1)}$ *and* $ab \in R_a$;

(iv) $ab \in S^{(1)}$ *and* $ab(ab)^{(1)}a = a$, *for some (eqivalently every)* $(ab)^{(1)} \in ab\{1\}$.

If these statements are true, then

$$a\{1\}_{bS} = \{bu \mid u \in S \text{ such that } a = abua\} = \{b(ab)^{(1)} \mid (ab)^{(1)} \in ab\{1\}\}.$$

Outer inverses in the prescribed Green's $\mathcal{R}$-class (cont.)

Theorem 3 – *The inner inverse in the prescribed principal right ideal*

The following statements are equivalent:

(i) *there exists an inner inverse of a contained in the principal right ideal $R(b)$;*

(ii) *there exists $u \in S$ such that $a = abua$;*

(iii) *$a \in S^{(1)}$ and $ab \in R_a$;*

(iv) *$ab \in S^{(1)}$ and $ab(ab)^{(1)}a = a$, for some (eqivalently every) $(ab)^{(1)} \in ab\{1\}$.*

If these statements are true, then

$$a\{1\}_{bS} = \{bu \mid u \in S \text{ such that } a = abua\} = \{b(ab)^{(1)} \mid (ab)^{(1)} \in ab\{1\}\}.$$

Theorem 3 – *The inner inverse in the prescribed principal right ideal*

The following statements are equivalent:

(i) *there exists an inner inverse of a contained in the principal right ideal $R(b)$;*

(ii) *there exists $u \in S$ such that $a = abua$;*

(iii) $a \in S^{(1)}$ *and* $ab \in R_a$;

(iv) $ab \in S^{(1)}$ *and* $ab(ab)^{(1)}a = a$, *for some (eqivalently every)* $(ab)^{(1)} \in ab\{1\}$.

If these statements are true, then

$$a\{1\}_{bS} = \{bu \mid u \in S \text{ such that } a = abua\} = \{b(ab)^{(1)} \mid (ab)^{(1)} \in ab\{1\}\}.$$

Outer inverses in the prescribed Green's $\mathcal{R}$-class (cont.)

Theorem 4 – *The $\{1,2\}$-inverse in the prescribed $\mathscr{R}$-class*

The following statements are equivalent:

(i) *there exists a $\{1,2\}$-inverse of a contained in the $\mathscr{R}$-class R_b;*

(ii) *there exist $u, v \in S$ such that $b = buab$ and $a = abva$;*

(iii) *there exists $w \in S$ such that $b = bwab$ and $a = abwa$;*

(iv) *there exist $s, t \in S$ such that $a = abs$ and $b = tab$;*

(v) *ab is a trace product;*

(vi) *$ab \in S^{(1)}$, $ab(ab)^{(1)}a = a$ and $b(ab)^{(1)}ab = b$, for some (eqivalently every) $(ab)^{(1)} \in ab\{1\}$.*

If these statements are true, then

$$a\{1,2\}_{R_b} = a\{2\}_{R_b} = a\{1\}_{R(b)}.$$

Trace product (F. Pastijn, 1982)

- the product ab is called a ***trace product*** if $ab \in R_a \cap L_b$
- ***Miller-Cliffords theorem***: $ab \in R_a \cap L_b \iff R_b \cap L_a$ *contains an idempotent*

Outer inverses in the prescribed Green's $\mathcal{R}$-class (cont.)

Theorem 4 – *The $\{1,2\}$-inverse in the prescribed $\mathscr{R}$-class*

The following statements are equivalent:

(i) *there exists a $\{1,2\}$-inverse of a contained in the $\mathscr{R}$-class R_b;*

(ii) *there exist $u, v \in S$ such that $b = buab$ and $a = abva$;*

(iii) *there exists $w \in S$ such that $b = bwab$ and $a = abwa$;*

(iv) *there exist $s, t \in S$ such that $a = abs$ and $b = tab$;*

(v) *ab is a trace product;*

(vi) *$ab \in S^{(1)}$, $ab(ab)^{(1)}a = a$ and $b(ab)^{(1)}ab = b$, for some (eqivalently every) $(ab)^{(1)} \in ab\{1\}$.*

If these statements are true, then

$$a\{1,2\}_{R_b} = a\{2\}_{R_b} = a\{1\}_{R(b)}.$$

Trace product (F. Pastijn, 1982)

- the product ab is called a ***trace product*** if $ab \in R_a \cap L_b$
- ***Miller-Cliffords theorem***: $ab \in R_a \cap L_b \iff R_b \cap L_a$ *contains an idempotent*

Outer inverses in the prescribed Green's $\mathcal{R}$-class (cont.)

Theorem 4 – *The $\{1,2\}$-inverse in the prescribed $\mathscr{R}$-class*

The following statements are equivalent:

(i) *there exists a $\{1,2\}$-inverse of a contained in the $\mathscr{R}$-class R_b;*

(ii) *there exist $u, v \in S$ such that $b = buab$ and $a = abva$;*

(iii) *there exists $w \in S$ such that $b = bwab$ and $a = abwa$;*

(iv) *there exist $s, t \in S$ such that $a = abs$ and $b = tab$;*

(v) *ab is a trace product;*

(vi) *$ab \in S^{(1)}$, $ab(ab)^{(1)}a = a$ and $b(ab)^{(1)}ab = b$, for some (eqivalently every) $(ab)^{(1)} \in ab\{1\}$.*

If these statements are true, then

$$a\{1,2\}_{R_b} = a\{2\}_{R_b} = a\{1\}_{R(b)}.$$

Trace product (F. Pastijn, 1982)

- the product ab is called a ***trace product*** if $ab \in R_a \cap L_b$
- ***Miller-Cliffords theorem***: $ab \in R_a \cap L_b \iff R_b \cap L_a$ *contains an idempotent*

Outer inverses in the prescribed Green's $\mathcal{R}$-class (cont.)

Theorem 4 – *The* $\{1,2\}$*-inverse in the prescribed* $\mathscr{R}$*-class*

The following statements are equivalent:

(i) *there exists a* $\{1,2\}$*-inverse of* a *contained in the* $\mathscr{R}$*-class* R_b*;*

(ii) *there exist* $u, v \in S$ *such that* $b = buab$ *and* $a = abva$*;*

(iii) *there exists* $w \in S$ *such that* $b = bwab$ *and* $a = abwa$*;*

(iv) *there exist* $s, t \in S$ *such that* $a = abs$ *and* $b = tab$*;*

(v) ab *is a trace product;*

(vi) $ab \in S^{(1)}$, $ab(ab)^{(1)}a = a$ *and* $b(ab)^{(1)}ab = b$*, for some (eqivalently every)* $(ab)^{(1)} \in ab\{1\}$.

If these statements are true, then

$$a\{1,2\}_{R_b} = a\{2\}_{R_b} = a\{1\}_{R(b)}.$$

Trace product (F. Pastijn, 1982)

- the product ab is called a ***trace product*** if $ab \in R_a \cap L_b$
- ***Miller-Cliffords theorem***: $ab \in R_a \cap L_b \iff R_b \cap L_a$ *contains an idempotent*

Outer inverses in the prescribed Green's $\mathcal{H}$-class

In the sequel, let S be a semigroup and let $a, d \in S$.

Mary's inverse along an element (Mary, 2011)

An element $x \in S$ is an ***inverse of*** a ***along*** d if any of the following two equivalent conditions holds

$$(M1)\quad xad = d = dax \ \text{ and } \ x \in R(d) \cap L(d), \qquad (M2)\quad xax = x \ \text{ and } \ x \,\mathscr{H}\, d.$$

(M2) means that an inverse of a along d is exactly an ***outer inverse of*** a ***in the Green's*** $\mathscr{H}$ ***-class*** H_d

Theorem – *X. Mary (LAA, 2011), X. Mary, P. Patrício (LAMA, 2013)*

The following statements are equivalent:

(i) *there exists an outer inverse of a contained in the $\mathscr{H}$-class H_d;*

(ii) *ad is group invertible and $ad \in L_d$;*

(iii) *da is group invertible and $da \in R_d$;*

(iv) *$dad \in H_d$.*

If these statements are true, then the inverse x of a along d is represented as follows:

$$x = d(ad)^{\#} = (da)^{\#}d.$$

Another approach

From now on, let S be a semigroup and $a, b, c \in S$.

Drazin's (b, c)-inverse (Drazin, LAA, 2012)

An element $x \in S$ is a (b, c)-***inverse*** of a if it satisfies

(a) $x \in bS \cap Sc$;

(b) $xab = b$ and $cax = c$.

If a has a (b, c)-inverse, then it is ***unique***, and in this case we say that a is (b, c)-***invertible***.

Theorem 5 – *The outer inverse in the prescribed $\mathscr{H}$-class*

The following two conditions for $x \in S$ are equivalent:

(i) *x is a (b, c)-inverse of a;*

(ii) *x is an outer inverse of a contained in the $\mathscr{H}$-class $R_b \cap L_c$.*

Drazin versus Mary

★ *Drazin's (b, c)-inverse ≡ Mary's inverse along d, for all triples b, c, d such that $R_b \cap L_c = H_d$.*

★ *the only difference –* ***in the way of representing Green's $\mathscr{H}$-classes****.*

Another approach

From now on, let S be a semigroup and $a, b, c \in S$.

Drazin's (b,c)-inverse (Drazin, LAA, 2012)

An element $x \in S$ is a (b,c)-***inverse*** of a if it satisfies

(a) $x \in bS \cap Sc$;

(b) $xab = b$ and $cax = c$.

If a has a (b,c)-inverse, then it is ***unique***, and in this case we say that a is (b,c)-***invertible***.

Theorem 5 – *The outer inverse in the prescribed $\mathscr{H}$-class*

The following two conditions for $x \in S$ are equivalent:

(i) *x is a (b,c)-inverse of a;*

(ii) *x is an outer inverse of a contained in the $\mathscr{H}$-class $R_b \cap L_c$.*

Drazin versus Mary

- *Drazin's (b,c)-inverse $\equiv$ Mary's inverse along d, for all triples b, c, d such that $R_b \cap L_c = H_d$.*
- *the only difference –* ***in the way of representing Green's $\mathscr{H}$-classes****.*

The existence of an outer inverse in the $\mathcal{H}$-class $R_b \cap L_c$

Theorem 6 – *The existence theorem*

The following statements are equivalent:

(i) *there exists an outer inverse of a contained in the $\mathscr{H}$-class $R_b \cap L_c$ (i.e., a (b, c)-inverse);*

(ii) $cab \in R_c \cap L_b$;

(iii) *cab is $\{1\}$-invertible, $cab(cab)^{(1)}c = c$ and $b(cab)^{(1)}cab = b$, for some (equiv. all) $(cab)^{(1)} \in cab\{1\}$;*

(iv) *there exist $u, v \in S$ such that $b = vcab$ and $c = cabu$;*

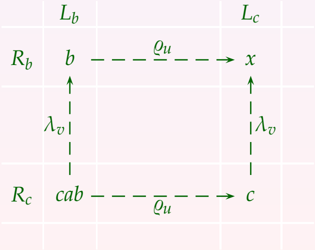
(v) *there exist $u, v \in S$ such that $b = bucab$ and $c = cabvc$;*

(vi) *there exists $w \in S$ such that $b = bwcab$ and $c = cabwc$;*

(vii) *there exist $u, v \in S$ such that $b = buab$, $c = cavc$ and $bu = vc$.*

- ⋆ (i)⇔(iv) – Drazin (2012)
- ⋆ (ii) – another way to write (iv)
- ⋆ the rest – new results

Visualization



The existence of an outer inverse in the $\mathcal{H}$-class $R_b \cap L_c$

Theorem 6 – *The existence theorem*

The following statements are equivalent:

(i) *there exists an outer inverse of a contained in the $\mathscr{H}$-class $R_b \cap L_c$ (i.e., a (b, c)-inverse);*

(ii) $cab \in R_c \cap L_b$;

(iii) *cab is $\{1\}$-invertible, $cab(cab)^{(1)}c = c$ and $b(cab)^{(1)}cab = b$, for some (equiv. all) $(cab)^{(1)} \in cab\{1\}$;*

(iv) *there exist $u, v \in S$ such that $b = vcab$ and $c = cabu$;*

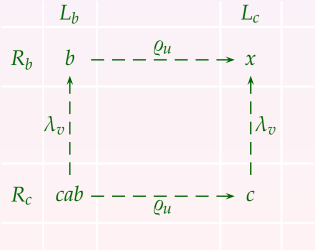
(v) *there exist $u, v \in S$ such that $b = bucab$ and $c = cabvc$;*

(vi) *there exists $w \in S$ such that $b = bwcab$ and $c = cabwc$;*

(vii) *there exist $u, v \in S$ such that $b = buab$, $c = cavc$ and $bu = vc$.*

- ★ (i)$\Leftrightarrow$(iv) – Drazin (2012)
- ★ (ii) – another way to write (iv)
- ★ the rest – new results

Visualization



The first representation theorem

Theorem 7 – *The first representation theorem*

If it exists, the outer inverse x of a contained in $R_b \cap L_c$ *can be represented as*

$$x = b(cab)^{(1)}c,$$

for an arbitrary $(cab)^{(1)} \in cab\{1\}$*, and it can also be represented as*

$$x = b(ab)^{(1)} = (ca)^{(1)}c,$$

for some $(ab)^{(1)} \in ab\{1\}$ *and* $(ca)^{(1)} \in ca\{1\}$*.*

The first representation theorem

Theorem 7 – *The first representation theorem*

If it exists, the outer inverse x of a contained in $R_b \cap L_c$ *can be represented as*

$$x = b(cab)^{(1)}c,$$

for an arbitrary $(cab)^{(1)} \in cab\{1\}$, *and it can also be represented as*

$$x = b(ab)^{(1)} = (ca)^{(1)}c,$$

for some $(ab)^{(1)} \in ab\{1\}$ *and* $(ca)^{(1)} \in ca\{1\}$.

Theorem 7 – *The first representation theorem*

If it exists, the outer inverse x of a contained in $R_b \cap L_c$ can be represented as

$$x = b(cab)^{(1)}c,$$

for an arbitrary $(cab)^{(1)} \in cab\{1\}$, and it can also be represented as

$$x = b(ab)^{(1)} = (ca)^{(1)}c,$$

for some $(ab)^{(1)} \in ab\{1\}$ and $(ca)^{(1)} \in ca\{1\}$.

Trace factorization

Theorem 8 – *The trace factorization theorem*

Let D be a $\mathscr{D}$-class of a semigroup S and $d \in D$. Then

(a) *For every $e \in D^\bullet$ there exist $u \in L_e$ and $v \in R_e$ such that $d = uv$, $R_d = R_u$ and $L_d = L_v$.*

(b) *For every pair $u, v \in S$ such that $d = uv$, $R_d = R_u$ and $L_d = L_v$ there exists $e \in D^\bullet$ such that $u \in L_e$ and $v \in R_e$.*

- ⋆ the representation $d = uv$
  with $e \in D^\bullet$, $u \in R_d \cap L_e$ and $v \in L_d \cap R_e$ –
  ***trace factorization of** d **with respect to** e*
  (since uv is a trace product)
- ⋆ direct generalization of the
  full-rank factorization of matrices
  d – matrix
  e – identity matrix of the same rank

	L_d		L_e
R_d	$d = uv$		u
R_e	v		e

Trace factorization

Theorem 8 – *The trace factorization theorem*

Let D be a $\mathscr{D}$-class of a semigroup S and $d \in D$. Then

(a) *For every $e \in D^\bullet$ there exist $u \in L_e$ and $v \in R_e$ such that $d = uv$, $R_d = R_u$ and $L_d = L_v$.*

(b) *For every pair $u, v \in S$ such that $d = uv$, $R_d = R_u$ and $L_d = L_v$ there exists $e \in D^\bullet$ such that $u \in L_e$ and $v \in R_e$.*

- ⋆ the representation $d = uv$

  with $e \in D^\bullet$, $u \in R_d \cap L_e$ and $v \in L_d \cap R_e$ –

  trace factorization of d with respect to e

  (since uv is a trace product)

- ⋆ direct generalization of the

  full-rank factorization of matrices

  d – matrix

  e – identity matrix of the same rank

	L_d		L_e	
R_d	$d = uv$		u	
R_e	v		e	

Trace factorization

Theorem 8 – *The trace factorization theorem*

Let D be a $\mathscr{D}$-class of a semigroup S and $d \in D$. Then

(a) *For every $e \in D^\bullet$ there exist $u \in L_e$ and $v \in R_e$ such that $d = uv$, $R_d = R_u$ and $L_d = L_v$.*

(b) *For every pair $u, v \in S$ such that $d = uv$, $R_d = R_u$ and $L_d = L_v$ there exists $e \in D^\bullet$ such that $u \in L_e$ and $v \in R_e$.*

- the representation $d = uv$
  with $e \in D^\bullet$, $u \in R_d \cap L_e$ and $v \in L_d \cap R_e$ –
  ***trace factorization of* d *with respect to* e**
  (since uv is a trace product)
- direct generalization of the
  full-rank factorization of matrices
  d – matrix
  e – identity matrix of the same rank

	L_d		L_e
R_d	$d = uv$		u
R_e	v		e

Trace factorization

Theorem 8 – *The trace factorization theorem*

Let D be a $\mathscr{D}$-class of a semigroup S and $d \in D$. Then

(a) *For every $e \in D^\bullet$ there exist $u \in L_e$ and $v \in R_e$ such that $d = uv$, $R_d = R_u$ and $L_d = L_v$.*

(b) *For every pair $u, v \in S$ such that $d = uv$, $R_d = R_u$ and $L_d = L_v$ there exists $e \in D^\bullet$ such that $u \in L_e$ and $v \in R_e$.*

- ⋆ the representation $d = uv$

  with $e \in D^\bullet$, $u \in R_d \cap L_e$ and $v \in L_d \cap R_e$ –

  ***trace factorization of** d **with respect to** e*

  (since uv is a trace product)

- ⋆ direct generalization of the

  full-rank factorization of matrices

  d – matrix

  e – identity matrix of the same rank

	L_d		L_e	
R_d	$d = uv$		u	
R_e	v		e	

Trace factorization

Theorem 8 – *The trace factorization theorem*

Let D be a $\mathscr{D}$-class of a semigroup S and $d \in D$. Then

(a) *For every $e \in D^\bullet$ there exist $u \in L_e$ and $v \in R_e$ such that $d = uv$, $R_d = R_u$ and $L_d = L_v$.*

(b) *For every pair $u, v \in S$ such that $d = uv$, $R_d = R_u$ and $L_d = L_v$ there exists $e \in D^\bullet$ such that $u \in L_e$ and $v \in R_e$.*

- ⋆ the representation $d = uv$

  with $e \in D^\bullet$, $u \in R_d \cap L_e$ and $v \in L_d \cap R_e$ –

  trace factorization of d with respect to e

  (since uv is a trace product)

- ⋆ direct generalization of the

  full-rank factorization of matrices

  d – matrix

  e – identity matrix of the same rank

	L_d		L_e
R_d	$d = uv$		u
R_e	v		e

Trace factorization

Theorem 8 – *The trace factorization theorem*

Let D be a $\mathscr{D}$-class of a semigroup S and $d \in D$. Then

(a) *For every $e \in D^{\bullet}$ there exist $u \in L_e$ and $v \in R_e$ such that $d = uv$, $R_d = R_u$ and $L_d = L_v$.*

(b) *For every pair $u, v \in S$ such that $d = uv$, $R_d = R_u$ and $L_d = L_v$ there exists $e \in D^{\bullet}$ such that $u \in L_e$ and $v \in R_e$.*

- ★ the representation $d = uv$

  with $e \in D^{\bullet}$, $u \in R_d \cap L_e$ and $v \in L_d \cap R_e$ –

  trace factorization of d with respect to e

  (since uv is a trace product)

- ★ direct generalization of the

  full-rank factorization of matrices

  d – matrix

  e – identity matrix of the same rank

	L_d		L_e
R_d	$d = uv$		u
R_e	v		e

The second representation theorem

Theorem 9

The following statements are equivalent:

(i) x *is a* (b, c)*-inverse of* a*;*

(ii) *for every* $e \in D^{\bullet}$ *there exist* $u \in L_e \cap R_b$ *and* $v \in R_e \cap L_c$ *such that* $vau \in H_e$ *and* $x = u(vau)^{\#}v$*;*

(iii) *there exist* $e \in D^{\bullet}$, $u \in L_e \cap R_b$ *and* $v \in R_e \cap L_c$ *such that* $vau \in H_e$ *and* $x = u(vau)^{\#}v$.

- ⋆ trace factorization

  of an arbitrary $d \in R_b \cap L_c$

	L_b		L_e		L_c	
R_b	b		u		x $d = uv$	
R_e			e vau		v	
R_c	cab				c	

The second representation theorem

Theorem 9

The following statements are equivalent:

(i) x *is a* (b, c)*-inverse of* a;

(ii) *for every* $e \in D^{\bullet}$ *there exist* $u \in L_e \cap R_b$ *and* $v \in R_e \cap L_c$ *such that* $vau \in H_e$ *and* $x = u(vau)^{\#}v$;

(iii) *there exist* $e \in D^{\bullet}$, $u \in L_e \cap R_b$ *and* $v \in R_e \cap L_c$ *such that* $vau \in H_e$ *and* $x = u(vau)^{\#}v$.

★ trace factorization

of an arbitrary $d \in R_b \cap L_c$

	L_b		L_e		L_c
R_b	b		u		x $d = uv$
R_e			e vau		v
R_c	cab				c

The second representation theorem

Theorem 9

The following statements are equivalent:

(i) x *is a* (b,c)*-inverse of* a;

(ii) *for every* $e \in D^\bullet$ *there exist* $u \in L_e \cap R_b$ *and* $v \in R_e \cap L_c$ *such that* $vau \in H_e$ *and* $x = u(vau)^\# v$;

(iii) *there exist* $e \in D^\bullet$, $u \in L_e \cap R_b$ *and* $v \in R_e \cap L_c$ *such that* $vau \in H_e$ *and* $x = u(vau)^\# v$.

★ trace factorization

of an arbitrary $d \in R_b \cap L_c$

	L_b		L_e		L_c
R_b	b		u		x $d = uv$
R_e			e vau		v
R_c	cab				c

The third representation theorem

Theorem 10

The following statements are equivalent:

(i) *x is a (b,c)-inverse of a;*

(ii) *for every $e \in D^\bullet$ there exist $u \in L_e \cap R_b$ and $v \in R_e \cap L_c$ such that $vau = e$ and $x = uv$;*

(iii) *there exist $e \in D^\bullet$, $u \in L_e \cap R_b$ and $v \in R_e \cap L_c$ such that $vau = e$ and $x = uv$.*

★ trace factorization of the (b,c)-inverse x

The third representation theorem

Theorem 10

The following statements are equivalent:

(i) x *is a* (b, c)*-inverse of* a;

(ii) *for every* $e \in D^\bullet$ *there exist* $u \in L_e \cap R_b$ *and* $v \in R_e \cap L_c$ *such that* $vau = e$ *and* $x = uv$;

(iii) *there exist* $e \in D^\bullet$, $u \in L_e \cap R_b$ *and* $v \in R_e \cap L_c$ *such that* $vau = e$ *and* $x = uv$.

- ⋆ trace factorization of the (b, c)-inverse x

$\{1,2\}$-inverses in prescribed Green's $\mathcal{H}$-classes

Theorem 11

The following statements are equivalent:

(i) *there exists a $\{1,2\}$-inverse of a contained in the $\mathscr{H}$-class $R_b \cap L_c$;*

(ii) *there exist a $\{1,2\}$-inverse of a contained in R_b and a $\{1,2\}$-inverse of a contained in L_c;*

(iii) *there exists $u \in S$ such that $b = bucab$, $c = cabuc$ and $a = abuca$.*

If these statements are true, the $\{1,2\}$-inverse x of a contained in the $\mathscr{H}$-class $R_b \cap L_c$ is represented by

$$x = buc = yaz,$$

for an arbitrary $u \in S$ such that $b = bucab$, and arbitrary $y \in a\{1,2\}_{R_b}$ and $z \in a\{1,2\}_{L_c}$.

$\{1,2\}$-inverses in prescribed Green's $\mathcal{H}$-classes

Theorem 11

The following statements are equivalent:

(i) *there exists a* $\{1,2\}$*-inverse of* a *contained in the* $\mathscr{H}$*-class* $R_b \cap L_c$*;*

(ii) *there exist a* $\{1,2\}$*-inverse of* a *contained in* R_b *and a* $\{1,2\}$*-inverse of* a *contained in* L_c*;*

(iii) *there exists* $u \in S$ *such that* $b = bucab$, $c = cabuc$ *and* $a = abuca$.

If these statements are true, the $\{1,2\}$*-inverse* x *of* a *contained in the* $\mathscr{H}$*-class* $R_b \cap L_c$ *is represented by*

$$x = buc = yaz,$$

for an arbitrary $u \in S$ *such that* $b = bucab$*, and arbitrary* $y \in a\{1,2\}_{R_b}$ *and* $z \in a\{1,2\}_{L_c}$.

$\{1,2\}$-inverses in prescribed Green's $\mathcal{H}$-classes

Theorem 11

The following statements are equivalent:

(i) *there exists a $\{1,2\}$-inverse of a contained in the $\mathscr{H}$-class $R_b \cap L_c$;*

(ii) *there exist a $\{1,2\}$-inverse of a contained in R_b and a $\{1,2\}$-inverse of a contained in L_c;*

(iii) *there exists $u \in S$ such that $b = bucab$, $c = cabuc$ and $a = abuca$.*

If these statements are true, the $\{1,2\}$-inverse x of a contained in the $\mathscr{H}$-class $R_b \cap L_c$ is represented by

$$x = buc = yaz,$$

for an arbitrary $u \in S$ such that $b = bucab$, and arbitrary $y \in a\{1,2\}_{R_b}$ and $z \in a\{1,2\}_{L_c}$.

$\{1,2\}$-inverses in prescribed Green's $\mathcal{H}$-classes

Theorem 11

The following statements are equivalent:

(i) *there exists a $\{1,2\}$-inverse of a contained in the $\mathscr{H}$-class $R_b \cap L_c$;*

(ii) *there exist a $\{1,2\}$-inverse of a contained in R_b and a $\{1,2\}$-inverse of a contained in L_c;*

(iii) *there exists $u \in S$ such that $b = bucab$, $c = cabuc$ and $a = abuca$.*

If these statements are true, the $\{1,2\}$-inverse x of a contained in the $\mathscr{H}$-class $R_b \cap L_c$ is represented by

$$x = buc = yaz,$$

for an arbitrary $u \in S$ such that $b = bucab$, and arbitrary $y \in a\{1,2\}_{R_b}$ and $z \in a\{1,2\}_{L_c}$.

Theorem 12

The following statements are equivalent:

(i) *there exists an outer inverse of a contained in the $\mathscr{R}$-class R_{a^*};*

(ii) *there exists an inner inverse of a contained in the principal right ideal $R(a^*)$;*

(iii) *there exists a $\{1,2\}$-inverse of a contained in the $\mathscr{R}$-class R_{a^*};*

(iv) *a is $\{1,4\}$-invertible;*

(v) *a is $\{1,2,4\}$-invertible;*

(vi) *there exists $u \in S$ such that $a^* = a^*uaa^*$;*

(vii) *there exists $v \in S$ such that $a^* = vaa^*$;*

(viii) *aa^* is a trace product.*

If these statements are true, then

$$a\{1,4\} = \{v \in S \mid a^* = vaa^*\},$$
$$a\{1,2,4\} = a\{2\}_{R_{a^*}} = a\{1\}_{R(a^*)} = a\{1,2\}_{R_{a^*}} = \{a^*u \mid u \in S \text{ such that } a^* = a^*uaa^*\}$$
$$= \{a^*(aa^*)^{(1)} \mid (aa^*)^{(1)} \in aa^*\{1\}\} = \{vav \mid v \in S \text{ such that } a^* = vaa^*\}.$$

Theorem 12

The following statements are equivalent:

(i) *there exists an outer inverse of a contained in the $\mathscr{R}$-class R_{a^*};*

(ii) *there exists an inner inverse of a contained in the principal right ideal $R(a^*)$;*

(iii) *there exists a $\{1,2\}$-inverse of a contained in the $\mathscr{R}$-class R_{a^*};*

(iv) *a is $\{1,4\}$-invertible;*

(v) *a is $\{1,2,4\}$-invertible;*

(vi) *there exists $u \in S$ such that $a^* = a^*uaa^*$;*

(vii) *there exists $v \in S$ such that $a^* = vaa^*$;*

(viii) *aa^* is a trace product.*

If these statements are true, then

$$a\{1,4\} = \{v \in S \mid a^* = vaa^*\},$$

$$a\{1,2,4\} = a\{2\}_{R_{a^*}} = a\{1\}_{R(a^*)} = a\{1,2\}_{R_{a^*}} = \{a^*u \mid u \in S \text{ such that } a^* = a^*uaa^*\}$$

$$= \{a^*(aa^*)^{(1)} \mid (aa^*)^{(1)} \in aa^*\{1\}\} = \{vav \mid v \in S \text{ such that } a^* = vaa^*\}.$$

Outer inverses in involutive semigroups (cont.)

Theorem 13

The following statements are equivalent:

(i) *there exists an outer inverse of a contained in the $\mathscr{H}$-class $R_{a^*} \cap L_{a^*}$;*

(ii) *there exist an outer inverse of a contained in R_{a^*} and an outer inverse of a contained in L_{a^*};*

(ii) *there exists an inner inverse of a contained in $R(a^*) \cap L(a^*)$;*

(iv) *there exists a $\{1,2\}$-inverse of a contained in the $\mathscr{H}$-class $R_{a^*} \cap L_{a^*}$;*

(v) *a is $\{1,3,4\}$-invertible;*

(vi) *a is MP-invertible;*

(vii) *there exists $u \in S$ such that $a^* = a^*aa^*u$;*

(viii) *there exists $v \in S$ such that $a^* = va^*aa^*$;*

(ix) *there exist $u, v \in S$ such that $a^* = a^*uaa^*$ and $a^* = a^*ava^*$;*

(x) *there exist $u, v \in S$ such that $a^* = vaa^*$ and $a^* = a^*au$;*

(xi) *a^*a and aa^* are trace products.*

If these statements are true, then

$$a^\dagger = a^*(a^*aa^*)^{(1)}a^* = (a^*a)^\# a^* = a^*(aa^*)^\# = a^*u = va^* = a^*paqa^* = sat,$$

for arbitrary $(a^*aa^*)^{(1)} \in a^*aa^*\{1\}$, $u \in S$ such that $a^* = a^*aa^*u$, $v \in S$ such that $a^* = va^*aa^*$, $p, q \in S$ such that $a^* = a^*paa^*$ and $a^* = a^*aqa^*$, and $s, t \in S$ such that $a^* = saa^*$ and $a^* = a^*at$.

Outer inverses in involutive semigroups (cont.)

Theorem 13

The following statements are equivalent:

(i) *there exists an outer inverse of a contained in the $\mathscr{H}$-class $R_{a^*} \cap L_{a^*}$;*

(ii) *there exist an outer inverse of a contained in R_{a^*} and an outer inverse of a contained in L_{a^*};*

(ii) *there exists an inner inverse of a contained in $R(a^*) \cap L(a^*)$;*

(iv) *there exists a $\{1,2\}$-inverse of a contained in the $\mathscr{H}$-class $R_{a^*} \cap L_{a^*}$;*

(v) *a is $\{1,3,4\}$-invertible;*

(vi) *a is MP-invertible;*

(vii) *there exists $u \in S$ such that $a^* = a^*aa^*u$;*

(viii) *there exists $v \in S$ such that $a^* = va^*aa^*$;*

(ix) *there exist $u, v \in S$ such that $a^* = a^*uaa^*$ and $a^* = a^*ava^*$;*

(x) *there exist $u, v \in S$ such that $a^* = vaa^*$ and $a^* = a^*au$;*

(xi) *a^*a and aa^* are trace products.*

If these statements are true, then

$$a^\dagger = a^*(a^*aa^*)^{(1)}a^* = (a^*a)^\# a^* = a^*(aa^*)^\# = a^*u = va^* = a^*paqa^* = sat,$$

for arbitrary $(a^*aa^*)^{(1)} \in a^*aa^*\{1\}$, $u \in S$ such that $a^* = a^*aa^*u$, $v \in S$ such that $a^* = va^*aa^*$, $p, q \in S$ such that $a^* = a^*paa^*$ and $a^* = a^*aqa^*$, and $s, t \in S$ such that $a^* = saa^*$ and $a^* = a^*at$.

Outer inverses in involutive semigroups (cont.)

Theorem 13

The following statements are equivalent:

(i) *there exists an outer inverse of a contained in the $\mathscr{H}$-class $R_{a^*} \cap L_{a^*}$;*

(ii) *there exist an outer inverse of a contained in R_{a^*} and an outer inverse of a contained in L_{a^*};*

(ii) *there exists an inner inverse of a contained in $R(a^*) \cap L(a^*)$;*

(iv) *there exists a $\{1,2\}$-inverse of a contained in the $\mathscr{H}$-class $R_{a^*} \cap L_{a^*}$;*

(v) *a is $\{1,3,4\}$-invertible;*

(vi) *a is MP-invertible;*

(vii) *there exists $u \in S$ such that $a^* = a^*aa^*u$;* **S. Crvenković (1982)**

(viii) *there exists $v \in S$ such that $a^* = va^*aa^*$;*

(ix) *there exist $u, v \in S$ such that $a^* = a^*uaa^*$ and $a^* = a^*ava^*$;*

(x) *there exist $u, v \in S$ such that $a^* = vaa^*$ and $a^* = a^*au$;*

(xi) *a^*a and aa^* are trace products.*

If these statements are true, then

$$a^\dagger = a^*(a^*aa^*)^{(1)}a^* = (a^*a)^\# a^* = a^*(aa^*)^\# = a^*u = va^* = a^*paqa^* = sat,$$

for arbitrary $(a^*aa^*)^{(1)} \in a^*aa^*\{1\}$, $u \in S$ such that $a^* = a^*aa^*u$, $v \in S$ such that $a^* = va^*aa^*$, $p, q \in S$ such that $a^* = a^*paa^*$ and $a^* = a^*aqa^*$, and $s, t \in S$ such that $a^* = saa^*$ and $a^* = a^*at$.

Outer inverses in involutive semigroups (cont.)

Theorem 13

The following statements are equivalent:

(i) *there exists an outer inverse of a contained in the $\mathscr{H}$-class $R_{a^*} \cap L_{a^*}$;*

(ii) *there exist an outer inverse of a contained in R_{a^*} and an outer inverse of a contained in L_{a^*};*

(ii) *there exists an inner inverse of a contained in $R(a^*) \cap L(a^*)$;*

(iv) *there exists a $\{1,2\}$-inverse of a contained in the $\mathscr{H}$-class $R_{a^*} \cap L_{a^*}$;*

(v) *a is $\{1,3,4\}$-invertible;*

(vi) *a is MP-invertible;*

(vii) *there exists $u \in S$ such that $a^* = a^*aa^*u$;*

(viii) *there exists $v \in S$ such that $a^* = va^*aa^*$;*

(ix) *there exist $u, v \in S$ such that $a^* = a^*uaa^*$ and $a^* = a^*ava^*$;*

(x) *there exist $u, v \in S$ such that $a^* = vaa^*$ and $a^* = a^*au$;*

(xi) *a^*a and aa^* are trace products.*

If these statements are true, then

$$a^\dagger = a^*(a^*aa^*)^{(1)}a^* = (a^*a)^\# a^* = a^*(aa^*)^\# = a^*u = va^* = a^*paqa^* = sat,$$

for arbitrary $(a^*aa^*)^{(1)} \in a^*aa^*\{1\}$, $u \in S$ such that $a^* = a^*aa^*u$, $v \in S$ such that $a^* = va^*aa^*$, $p, q \in S$ such that $a^* = a^*paa^*$ and $a^* = a^*aqa^*$, and $s, t \in S$ such that $a^* = saa^*$ and $a^* = a^*at$.

Outer inverses in involutive semigroups (cont.)

Theorem 14

Let S be an involutive semigroup, let $a \in S$ be an MP-invertible element, let D be the $\mathscr{D}$-class of S containing a and a^*.

Let $a^* = uv$ be a trace factorization of a^* with respect to an arbitrary $e \in D^\bullet$. Then $vau \in H_e$ and

$$a^\dagger = u(vau)^\# v.$$

Applications

Generalized inverses of complex matrices

- ***P. Stanimirović, M. Ćirić, I. Stojanović, D. Gerontitis***, *Conditions for existence, representations and computation of matrix generalized inverses*, COMPLEXITY Vol. 2017 (2017) Article ID 6429725, 27 pages.

  computation based on representations in terms of linear equations
  and the well-known method for solving matrix equations of the form $AXB = D$

  existence criteria – given in terms of ranks (e.g., $\operatorname{rank}(CAB) = \operatorname{rank}(B) = \operatorname{rank}(C)$)

Generalized inverses of fuzzy matrices

- ***M. Ćirić, J. Ignjatović***, *The existence of generalized inverses of fuzzy matrices*, in: L. Kóczy, J. Kacprzyk, J. Medina (eds.), ESCIM 2016, STUDIES IN COMPUTATIONAL INTELLIGENCE, Springer 2017, to appear.

  computation based on our methods for solving equations and inequalities
  defined by residuated functions

Generalized inverses in residuated semigroups and quantales

- ***J. Ignjatović, M. Ćirić***, *Moore-Penrose equations in involutive residuated semigroups and involutive quantales*, FILOMAT 31:2 (2017) 183–196.

  a similar methodology