A semigroup-theoretical approach to the study of generalized inverses

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joint work with
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### Origins of generalized inverses

#### Historical notes

- **E. I. Fredholm** (1903) – generalized inverses of integral operators;
- **D. Hilbert** (1904) – differential operators, **A. Hurwitz** (1912), and others;
- **E. H. Moore** (1920 or earlier) – generalized inverses of matrices (**general reciprocal**);
- Moore’s work has not attracted more attention until the 1950s;
- **A. Bjerhammar** (1951) – links with solutions of linear systems;
  - least-squares solutions (**approximate**) and **minimum-norm solutions**;
- **R. Penrose** (1955) – generalized inverses as solutions of algebraic equations;
  - **Moore-Penrose equations**;
  - in abstract algebraic structures – semigroups, rings, Banach algebras, $C^*$-algebras, etc.;

#### Applications

- solving matrix equations;
- solving singular differential and difference equations;
- the investigation of Cesaro-Neumann iterations;
- the least squares approximation;
- finite Markov chains, cryptography, statistics, etc.
Moore-Penrose equations

★ $S$ – an involution semigroup;
★ $a \in S$ – fixed element, $x$ – unknown taking values in $S$;

★ *Moore-Penrose equations*:

(1) $axa = a$
(2) $xax = x$
(3) $(ax)^* = ax$
(4) $(xa)^* = xa$

★ Additionally, we consider the equation

(5) $ax = xa$

γ-inverses

★ for $\gamma \subseteq \{1, 2, 3, 4, 5\}$, by $\langle \gamma \rangle$ we denote the system consisting of the equations $(i)$, for all $i \in \gamma$;
★ if $\langle \gamma \rangle$ has a solution, then $a$ is called $\gamma$-invertible.
★ in this case, solutions of $\langle \gamma \rangle$ are called $\gamma$-inverses of $a$. 
## Terminology and notation

### Terminology

<table>
<thead>
<tr>
<th>Inverse Type</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>({1})-inverse</td>
<td><em>g-inverse</em> (&quot;generalized inverse&quot;) or <em>inner inverse</em></td>
</tr>
<tr>
<td>({2})-inverse</td>
<td><em>outer inverse</em></td>
</tr>
<tr>
<td>({1,2})-inverse</td>
<td><em>reflexive g-inverse</em> or <em>Thierrin-Vagner inverse</em></td>
</tr>
<tr>
<td>({1,3})-inverse</td>
<td><em>last-squares g-inverse</em></td>
</tr>
<tr>
<td>({1,4})-inverse</td>
<td><em>minimum-norm g-inverse</em></td>
</tr>
<tr>
<td>({1,2,3,4})-inverse</td>
<td><em>Moore-Penrose inverse</em> or <em>MP-inverse</em></td>
</tr>
<tr>
<td>({1,2,5})-inverse</td>
<td><em>group inverse</em></td>
</tr>
</tbody>
</table>

* When exist, the Moore-Penrose inverse and the group inverse of *a* are *unique*.

### Notation

<table>
<thead>
<tr>
<th>Symbol</th>
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<tbody>
<tr>
<td><em>a</em>(^\dagger)</td>
<td>Moore-Penrose inverse of <em>a</em></td>
</tr>
<tr>
<td><em>a</em>(^#)</td>
<td>Group inverse of <em>a</em></td>
</tr>
<tr>
<td><em>a</em>(\gamma)</td>
<td>The set of all <em>γ</em>-inverses of <em>a</em></td>
</tr>
<tr>
<td><em>a</em>(\gamma)_(X)</td>
<td>The set of all <em>γ</em>-inverses of <em>a</em> contained in the set <em>X</em></td>
</tr>
<tr>
<td><em>S</em>((1))</td>
<td>The set of all ({1})-invertible elements *(Reg(<em>S</em>))</td>
</tr>
<tr>
<td><em>X</em>(^\bullet)</td>
<td>The set of all idempotents contained in the set <em>X</em></td>
</tr>
</tbody>
</table>
Matrices with prescribed range and null-space

<table>
<thead>
<tr>
<th>Range, null space, rank</th>
</tr>
</thead>
<tbody>
<tr>
<td>★ $A \in \mathbb{C}^{m \times n}$ – a complex matrix of type $m \times n$;</td>
</tr>
<tr>
<td>★ range or image of $A$</td>
</tr>
<tr>
<td>$R(A) = {y \in \mathbb{C}^{m} \mid Ax = y, \text{ for some } x \in \mathbb{C}^{n}}$</td>
</tr>
<tr>
<td>★ null space or kernel of $A$</td>
</tr>
<tr>
<td>$N(A) = {x \in \mathbb{C}^{n} \mid Ax = 0}$</td>
</tr>
<tr>
<td>★ rank($A$) – the rank of $A$ (the dimension of the column space and the row space of $A$).</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Generalized inverses with prescribed range and null space</th>
</tr>
</thead>
<tbody>
<tr>
<td>★ $T \subseteq \mathbb{C}^{m}$, $S \subseteq \mathbb{C}^{n}$ – given subspaces, $A \in \mathbb{C}^{m \times n}$;</td>
</tr>
<tr>
<td>★ $A^{(2)}_{T,S}$ – the ${2}$-inverse of $A$ with prescribed range $T$ and null space $S$ (if it exists)</td>
</tr>
<tr>
<td>★ $A^{(1,2)}_{T,S}$ – the ${1,2}$-inverse of $A$ with prescribed range $T$ and null space $S$ (if it exists)</td>
</tr>
<tr>
<td>$A^{\dagger} = A^{(2)}<em>{R(A^<em>),N(A^</em>)} = A^{(1,2)}</em>{R(A^<em>),N(A^</em>)}$</td>
</tr>
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<td>$A^{#} = A^{(2)}<em>{R(A),N(A)} = A^{(1,2)}</em>{R(A),N(A)}$</td>
</tr>
<tr>
<td>★ $A^*$ – the conjugate transpose of $A$</td>
</tr>
</tbody>
</table>
Ring-theoretical generalizations

Generalized inverses with prescribed idempotents

★ D. Djordjević, Y. Wei, Communications in Algebra 33 (2005) 3051–3060

outer inverses with prescribed idempotents (\(xax = x\) and idempotents \(ax, xa\) are prescribed)

★ B. Načevska, D. Djordjević, Communications in Algebra 39 (2011) 634–646

inner inverses with prescribed idempotents (\(a = axa\) and idempotents \(ax, xa\) are prescribed)

Generalized inverses with prescribed ideals


outer inverses with prescribed kernel ideals (left and right annihilators)


outer inverses with prescribed image ideals (principal left and right ideals)
Semigroup of matrices $M_\varnothing(\mathbb{C})$

Let $M(\mathbb{C})$ be the set of all matrices of any type with entries in $\mathbb{C}$, i.e.,

$$M(\mathbb{C}) = \bigcup_{m,n \in \mathbb{N}} \mathbb{C}^{m \times n},$$

Let $M_\varnothing(\mathbb{C}) = M(\mathbb{C}) \cup \{\varnothing\}$, where $\varnothing \notin M(\mathbb{C})$.

For the sake of convenience, we call $\varnothing$ the empty matrix.

The multiplication in $M_\varnothing(\mathbb{C})$ is defined using the standard procedure for converting a partial semigroup into a semigroup:

- the product in $M_\varnothing(\mathbb{C})$ coincides with the ordinary matrix product, whenever it is defined;
- in all other cases the product is equal to the empty matrix $\varnothing$.

With respect to this multiplication, $M_\varnothing(\mathbb{C})$ is a semigroup with the zero $\varnothing$.

We call $M_\varnothing(\mathbb{C})$ the semigroup of matrices with entries in $\mathbb{C}$.
Green’s equivalences in the semigroup of matrices

For any \( A, B \in M(C) \) we have

\[
A R B \Leftrightarrow R(A) = R(B),
\]

\[
A L B \Leftrightarrow N(A) = N(B),
\]

\[
A H B \Leftrightarrow R(A) = R(B) \& N(A) = N(B),
\]

\[
A D B \Leftrightarrow \text{rank}(A) = \text{rank}(B)
\]

On the other hand,

\[
D_\emptyset = R_\emptyset = L_\emptyset = H_\emptyset = \{\emptyset\}.
\]

Our mission

Outer inverses belonging to prescribed Green’s equivalence classes

- M. Ćirić, P. Stanimirović, J. Ignjatović, Outer inverses in semigroups belonging to prescribed Green’s equivalence classes, to appear.
In the sequel, let $S$ be a semigroup and let $a, b \in S$.

**Theorem 1**

An element $x \in S$ is an outer inverse of $a$ contained in the $R$-class $R_b$ if and only if 

$$x \in bS \quad \text{and} \quad xab = b.$$ 

**Theorem 2 – The outer inverse in the prescribed $R$-class**

The following statements are equivalent:

(i) there exists an outer inverse of $a$ contained in the $R$-class $R_b$;

(ii) there exists $u \in S$ such that $b = buab$;

(iii) $b \in S^{(1)}$ and $ab \in L_b$;

(iv) $ab \in S^{(1)}$ and $b(ab)^{(1)}ab = b$, for some (equivalently every) $(ab)^{(1)} \in ab\{1\}$.

If these statements are true, then 

$$a\{2\}R_b = \{bu \mid u \in S \text{ such that } b = buab\} = \{b(ab)^{(1)} \mid (ab)^{(1)} \in ab\{1\}\}.$$
In the sequel, let $S$ be a semigroup and let $a, b \in S$.

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Theorem 3 – *The inner inverse in the prescribed principal right ideal*

The following statements are equivalent:

(i) there exists an inner inverse of \( a \) contained in the principal right ideal \( R(b) \);

(ii) there exists \( u \in S \) such that \( a = abua \);

(iii) \( a \in S^{(1)} \) and \( ab \in R_a \);

(iv) \( ab \in S^{(1)} \) and \( ab(ab)^{(1)}a = a \), for some (equivalently every) \( (ab)^{(1)} \in ab{1} \).

If these statements are true, then

\[
a{1}_bS = \{bu \mid u \in S \text{ such that } a = abua\} = \{b(ab)^{(1)} \mid (ab)^{(1)} \in ab{1}\}.
\]
Theorem 3 – *The inner inverse in the prescribed principal right ideal*

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\]
Theorem 3 – The inner inverse in the prescribed principal right ideal

The following statements are equivalent:

(i) there exists an inner inverse of a contained in the principal right ideal $R(b)$;

(ii) there exists $u \in S$ such that $a = abua$;

(iii) $a \in S^{(1)}$ and $ab \in R_a$;

(iv) $ab \in S^{(1)}$ and $ab(ab)^{(1)}a = a$, for some (equivalently every) $(ab)^{(1)} \in ab\{1\}$.

If these statements are true, then

$$a\{1\}_bS = \{bu \mid u \in S \text{ such that } a = abua\} = \{b(ab)^{(1)} \mid (ab)^{(1)} \in ab\{1\}\}.$$

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Semigroup-theoretical approach to generalized inverses
Theorem 4 – The \{1, 2\}-inverse in the prescribed $\mathcal{R}$-class

The following statements are equivalent:

(i) there exists a \{1, 2\}-inverse of $a$ contained in the $\mathcal{R}$-class $R_b$;

(ii) there exist $u, v \in S$ such that $b = buab$ and $a = abva$;

(iii) there exists $w \in S$ such that $b = bwab$ and $a = abwa$;

(iv) there exist $s, t \in S$ such that $a = abs$ and $b = tab$;

(v) $ab$ is a trace product;

(vi) $ab \in S^{(1)}$, $ab(ab)^{(1)}a = a$ and $b(ab)^{(1)}ab = b$, for some (equivalently every) $(ab)^{(1)} \in ab \{1\}$.

If these statements are true, then

$$a \{1, 2\} \mathcal{R}_b = a \{2\} \mathcal{R}_b = a \{1\} \mathcal{R}_{\mathcal{R}(b)}.$$ 

Trace product (F. Pastijn, 1982)

- the product $ab$ is called a **trace product** if $ab \in R_a \cap L_b$
- **Miller-Cliffords theorem**: $ab \in R_a \cap L_b \iff R_b \cap L_a$ contains an idempotent
Theorem 4 – The \( \{1, 2\} \)-inverse in the prescribed \( R \)-class

The following statements are equivalent:

(i) there exists a \( \{1, 2\} \)-inverse of \( a \) contained in the \( R \)-class \( R_b \);

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If these statements are true, then

\[ a\{1, 2\}_R b = a\{2\}_R b = a\{1\}_{R(b)} . \]

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Theorem 4 – The \{1, 2\}-inverse in the prescribed R-class

The following statements are equivalent:

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(iii) there exists \(w \in S\) such that \(b = bwab\) and \(a = abwa\);

(iv) there exist \(s, t \in S\) such that \(a = abs\) and \(b = tab\);

(v) \(ab\) is a trace product;

(vi) \(ab \in S^{(1)}\), \(ab(ab)^{(1)}a = a\) and \(b(ab)^{(1)}ab = b\), for some (equivalently every) \((ab)^{(1)} \in ab\{1\}\).

If these statements are true, then

\[ a\{1, 2\}_R = a\{2\}_R = a\{1\}_{R(b)}. \]

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The following statements are equivalent:

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If these statements are true, then

$$a\{1, 2\}R_b = a\{2\}R_b = a\{1\}R(b).$$

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- the product $ab$ is called a *trace product* if $ab \in R_a \cap L_b$
- *Miller-Cliffords theorem*: $ab \in R_a \cap L_b \iff R_b \cap L_a$ contains an idempotent
In the sequel, let $S$ be a semigroup and let $a, d \in S$.

**Mary’s inverse along an element (Mary, 2011)**

An element $x \in S$ is an *inverse of $a$ along $d$* if any of the following two equivalent conditions holds:

\[(M1) \quad xad = d = dax \quad \text{and} \quad x \in R(d) \cap L(d), \quad (M2) \quad xax = x \quad \text{and} \quad x \mathcal{H} d.\]

(M2) means that an inverse of $a$ along $d$ is exactly an *outer inverse of $a$ in the Green’s $\mathcal{H}$-class $H_d$*.


The following statements are equivalent:

(i) there exists an outer inverse of $a$ contained in the $\mathcal{H}$-class $H_d$;

(ii) $ad$ is group invertible and $ad \in L_d$;

(iii) $da$ is group invertible and $da \in R_d$;

(iv) $dad \in H_d$.

If these statements are true, then the inverse $x$ of $a$ along $d$ is represented as follows:

\[x = d(ad)^\# = (da)^\# d.\]
Another approach

From now on, let $S$ be a semigroup and $a, b, c \in S$.

**Drazin’s $(b, c)$-inverse (Drazin, LAA, 2012)**

An element $x \in S$ is a $(b, c)$-inverse of $a$ if it satisfies

(a) $x \in bS \cap Sc$;

(b) $xab = b$ and $cax = c$.

If $a$ has a $(b, c)$-inverse, then it is unique, and in this case we say that $a$ is $(b, c)$-invertible.

**Theorem 5 – The outer inverse in the prescribed $H$-class**

The following two conditions for $x \in S$ are equivalent:

(i) $x$ is a $(b, c)$-inverse of $a$;

(ii) $x$ is an outer inverse of $a$ contained in the $H$-class $R_b \cap L_c$.

**Drazin versus Mary**

★ Drazin’s $(b, c)$-inverse $\equiv$ Mary’s inverse along $d$, for all triples $b, c, d$ such that $R_b \cap L_c = H_d$.

★ the only difference – in the way of representing Green’s $H$-classes.
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- **the only difference – in the way of representing Green’s $\mathcal{H}$-classes.**
The existence of an outer inverse in the $\mathcal{H}$-class $R_b \cap L_c$

**Theorem 6 – The existence theorem**

The following statements are equivalent:

(i) there exists an outer inverse of $a$ contained in the $\mathcal{H}$-class $R_b \cap L_c$ (i.e., a $(b, c)$-inverse);

(ii) $cab \in R_c \cap L_b$;

(iii) $cab$ is $\{1\}$-invertible, $cab(cab)^{(1)}c = c$ and $b(cab)^{(1)}cab = b$, for some (equiv. all) $(cab)^{(1)} \in cab\{1\}$;

(iv) there exist $u, v \in S$ such that $b = vcab$ and $c = cabu$;

(v) there exist $u, v \in S$ such that $b = bucab$ and $c = cabvc$;

(vi) there exists $w \in S$ such that $b = bwcab$ and $c = cabwc$;

(vii) there exist $u, v \in S$ such that $b = buab$, $c = cavc$ and $bu = vc$.

★ (i)$\Leftrightarrow$(iv) – Drazin (2012)

★ (ii) – another way to write (iv)

★ the rest – new results

Visualization

$$
\begin{array}{ccc}
& L_b & \quad & L_c \\
R_b & b & \xrightarrow{Q_{tu}} & x \\
\lambda_v & \quad & \lambda_v \\
R_c & cab & \xrightarrow{Q_{tu}} & c \\
\end{array}
$$
The existence of an outer inverse in the $\mathcal{H}$-class $R_b \cap L_c$

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**Visualization**

\[
\begin{array}{c|c|c}
 & L_b & L_c \\
\hline
R_b & b & Q_u x \\
\lambda_v & | & \\
R_c & cab & Q_u c \\
\end{array}
\]
The first representation theorem

**Theorem 7 – The first representation theorem**

If it exists, the outer inverse $x$ of $a$ contained in $R_b \cap L_c$ can be represented as

$$x = b(cab)^{(1)} c,$$

for an arbitrary $(cab)^{(1)} \in cab\{1\}$, and it can also be represented as

$$x = b(ab)^{(1)} = (ca)^{(1)} c,$$

for some $(ab)^{(1)} \in ab\{1\}$ and $(ca)^{(1)} \in ca\{1\}$. 
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Theorem 8 – The trace factorization theorem

Let $D$ be a $D$-class of a semigroup $S$ and $d \in D$. Then

(a) For every $e \in D^\bullet$ there exist $u \in L_e$ and $v \in R_e$ such that $d = uv$, $R_d = R_u$ and $L_d = L_v$.

(b) For every pair $u, v \in S$ such that $d = uv$, $R_d = R_u$ and $L_d = L_v$ there exists $e \in D^\bullet$ such that $u \in L_e$ and $v \in R_e$.

★ the representation $d = uv$

with $e \in D^\bullet$, $u \in R_d \cap L_e$ and $v \in L_d \cap R_e$ –

\textit{trace factorization of $d$ with respect to $e$}

(since $uv$ is a trace product)

★ direct generalization of the

\textit{full-rank factorization of matrices}

$d$ – matrix

$e$ – identity matrix of the same rank
Theorem 8 – The trace factorization theorem

Let \( D \) be a \( \mathcal{D} \)-class of a semigroup \( S \) and \( d \in D \). Then

(a) For every \( e \in D^\bullet \) there exist \( u \in L_e \) and \( v \in R_e \) such that \( d = uv \), \( R_d = R_u \) and \( L_d = L_v \).

(b) For every pair \( u, v \in S \) such that \( d = uv \), \( R_d = R_u \) and \( L_d = L_v \) there exists \( e \in D^\bullet \) such that \( u \in L_e \) and \( v \in R_e \).

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| \( L_d \) | \( L_e \) |
| \( R_d \) | \( d = uv \) | \( u \) |
| \( R_e \) | \( v \) | \( e \) |
Theorem 8 – The trace factorization theorem

Let $D$ be a $\mathbb{D}$-class of a semigroup $S$ and $d \in D$. Then

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★ direct generalization of the

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$d$ – matrix

$e$ – identity matrix of the same rank
The second representation theorem

**Theorem 9**

The following statements are equivalent:

(i) \( x \) is a \((b,c)\)-inverse of \( a \);

(ii) for every \( e \in D^\bullet \) there exist \( u \in L_e \cap R_b \) and \( v \in R_e \cap L_c \) such that \( vau \in H_e \) and \( x = u(vau)^\# v \);

(iii) there exist \( e \in D^\bullet \), \( u \in L_e \cap R_b \) and \( v \in R_e \cap L_c \) such that \( vau \in H_e \) and \( x = u(vau)^\# v \).

★ trace factorization

of an arbitrary \( d \in R_b \cap L_c \)

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Semigroup-theoretical approach to generalized inverses
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The following statements are equivalent:

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★ trace factorization

d of an arbitrary \( d \in R_b \cap L_c \)

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The third representation theorem

**Theorem 10**

The following statements are equivalent:

(i) $x$ is a $(b,c)$-inverse of $a$;

(ii) for every $e \in D^\bullet$ there exist $u \in L_e \cap R_b$ and $v \in R_e \cap L_c$ such that $vau = e$ and $x = uv$;

(iii) there exist $e \in D^\bullet$, $u \in L_e \cap R_b$ and $v \in R_e \cap L_c$ such that $vau = e$ and $x = uv$.

★ trace factorization of the $(b,c)$-inverse $x$
The third representation theorem

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(iii) there exist \( e \in D^\bullet \), \( u \in L_e \cap R_b \) and \( v \in R_e \cap L_c \) such that \( vau = e \) and \( x = uv \).

★ trace factorization of the \((b,c)\)-inverse \( x \)
Theorem 11

The following statements are equivalent:

(i) there exists a \{1, 2\}-inverse of \(a\) contained in the \(H\)-class \(R_b \cap L_c\);

(ii) there exist a \{1, 2\}-inverse of \(a\) contained in \(R_b\) and a \{1, 2\}-inverse of \(a\) contained in \(L_c\);

(iii) there exists \(u \in S\) such that \(b = bucab\), \(c = cabuc\) and \(a = abuca\).

If these statements are true, the \{1, 2\}-inverse \(x\) of \(a\) contained in the \(H\)-class \(R_b \cap L_c\) is represented by

\[ x = buc = yaz, \]

for an arbitrary \(u \in S\) such that \(b = bucab\), and arbitrary \(y \in a\{1, 2\}_R\) and \(z \in a\{1, 2\}_L\).
Theorem 11

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The following statements are equivalent:

(i) there exists a \{1,2\}-inverse of a contained in the $\mathcal{H}$-class $R_b \cap L_c$;

(ii) there exist a \{1,2\}-inverse of a contained in $R_b$ and a \{1,2\}-inverse of a contained in $L_c$;

(iii) there exists $u \in S$ such that $b = bucab$, $c = cabuc$ and $a = abuca$.

If these statements are true, the \{1,2\}-inverse $x$ of a contained in the $\mathcal{H}$-class $R_b \cap L_c$ is represented by

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for an arbitrary $u \in S$ such that $b = bucab$, and arbitrary $y \in a\{1,2\}_R$ and $z \in a\{1,2\}_L$. 

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{1,2}-inverses in prescribed Green’s $\mathcal{H}$-classes

Theorem 11
The following statements are equivalent:

(i) there exists a \{1, 2\}-inverse of a contained in the \(H\)-class \(R_b \cap L_c\);

(ii) there exist a \{1, 2\}-inverse of a contained in \(R_b\) and a \{1, 2\}-inverse of a contained in \(L_c\);

(iii) there exists \(u \in S\) such that \(b = buca\), \(c = cabu\) and \(a = abuc\).

If these statements are true, the \{1, 2\}-inverse \(x\) of a contained in the \(H\)-class \(R_b \cap L_c\) is represented by

\[ x = buca = yaz, \]

for an arbitrary \(u \in S\) such that \(b = buca\), and arbitrary \(y \in a\{1, 2\}_{R_b}\) and \(z \in a\{1, 2\}_{L_c}\).
Outer inverses in involutive semigroups

Theorem 12

The following statements are equivalent:

(i) there exists an outer inverse of $a$ contained in the $\mathcal{R}$-class $R_{a^*}$;

(ii) there exists an inner inverse of $a$ contained in the principal right ideal $R(a^*)$;

(iii) there exists a $\{1, 2\}$-inverse of $a$ contained in the $\mathcal{R}$-class $R_{a^*}$;

(iv) $a$ is $\{1, 4\}$-invertible;

(v) $a$ is $\{1, 2, 4\}$-invertible;

(vi) there exists $u \in S$ such that $a^* = a^* u a^*$;

(vii) there exists $v \in S$ such that $a^* = v a^*$;

(viii) $a a^*$ is a trace product.

If these statements are true, then

\[ a\{1, 4\} = \{v \in S \mid a^* = v a^*\}, \]

\[ a\{1, 2, 4\} = a\{2\}_{R_{a^*}} = a\{1\}_{R(a^*)} = a\{1, 2\}_{R_{a^*}} = \{a^* u \mid u \in S \text{ such that } a^* = a^* u a^*\} \]

\[ = \{a^* (aa^*)^{(1)} \mid (aa^*)^{(1)} \in aa^*\{1\}\} = \{v a v \mid v \in S \text{ such that } a^* = v a^*\}. \]
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a\{1, 4\} = \{v \in S \mid a^* = v a a^*\},\\
a\{1, 2, 4\} = a\{2\}_{R_{a^*}} = a\{1\}_{R(a^*)} = a\{1, 2\}_{R_{a^*}} = \{a^* u \mid u \in S \text{ such that } a^* = a^* u a a^*\} = \{a^*(a a^*)^{(1)} \mid (a a^*)^{(1)} \in a a^*\{1\}\} = \{v a v \mid v \in S \text{ such that } a^* = v a a^*\}.
\]
The following statements are equivalent:

(i) there exists an outer inverse of $a$ contained in the $\mathcal{H}$-class $R_{a*} \cap L_{a*}$;
(ii) there exist an outer inverse of $a$ contained in $R_{a*}$ and an outer inverse of $a$ contained in $L_{a*}$;
(iii) there exists an inner inverse of $a$ contained in $R(a*) \cap L(a*)$;
(iv) there exists a $\{1, 2\}$-inverse of $a$ contained in the $\mathcal{H}$-class $R_{a*} \cap L_{a*}$;
(v) $a$ is $\{1, 3, 4\}$-invertible;
(vi) $a$ is MP-invertible;
(vii) there exists $u \in S$ such that $a^* = a^*aa^* u$;
(viii) there exists $v \in S$ such that $a^* = va^*aa^*$;
(ix) there exist $u, v \in S$ such that $a^* = a^*uaa^*$ and $a^* = a^*ava^*$;
(x) there exist $u, v \in S$ such that $a^* = vaa^*$ and $a^* = a^*au$;
(xi) $a^*a$ and $aa^*$ are trace products.

If these statements are true, then

$$a^\dagger = a^*(a^*aa^*)^{(1)}a^* = (a^* a)^\#a^* = a^*(aa^*)^\# = a^* u = va^* = a^* p a q a^* = sat,$$

for arbitrary $(a^*aa^*)^{(1)} \in a^*aa^* \{1\}$, $u \in S$ such that $a^* = a^*aa^* u$, $v \in S$ such that $a^* = va^*aa^*$, $p, q \in S$ such that $a^* = a^*pa a^*$ and $a^* = a^*qa a^*$, and $s, t \in S$ such that $a^* = sa a^*$ and $a^* = a^* at$. 
Theorem 13

The following statements are equivalent:

(i) there exists an outer inverse of a contained in the $\mathcal{H}$-class $R_{a^*} \cap L_{a^*}$;
(ii) there exist an outer inverse of a contained in $R_{a^*}$ and an outer inverse of a contained in $L_{a^*}$;
(ii) there exists an inner inverse of a contained in $R(a^*) \cap L(a^*)$;
(iv) there exists a $\{1, 2\}$-inverse of a contained in the $\mathcal{H}$-class $R_{a^*} \cap L_{a^*}$;
(v) a is $\{1, 3, 4\}$-invertible;
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(ii) there exists an inner inverse of $a$ contained in $R(a^*) \cap L(a^*)$;
(iv) there exists a $\{1, 2\}$-inverse of $a$ contained in the $H$-class $R_a^* \cap L_a^*$;
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S. Crvenković (1982)
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for arbitrary $(a^*aa^*)^{(1)} \in a^*aa^*\{1\}$, $u \in S$ such that $a^* = a^*aa^*u$, $v \in S$ such that $a^* = va^*aa^*$, $p, q \in S$ such that $a^* = a^*paa^*$ and $a^* = a^*aqa^*$, and $s, t \in S$ such that $a^* = saa^*$ and $a^* = a^*at$. 

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(x) there exist \(u, v \in S\) such that \(a^* = va^*a^*\) and \(a^* = a^*au\);

(xi) \(a^*a\) and \(aa^*\) are trace products.

If these statements are true, then

\[
a^\dagger = a^*(a^*aa^*)^{(1)}a^* = (a^*a)^#a^* = a^*(aa^*)^# = a^*u = va^* = a^*paqa^* = sat,
\]

for arbitrary \((a^*aa^*)^{(1)} \in a^*aa^*\{1\}, u \in S\) such that \(a^* = a^*aa^*u\), \(v \in S\) such that \(a^* = va^*aa^*\), \(p, q \in S\) such that \(a^* = a^*pa^*\) and \(a^* = a^*qa^*\), and \(s, t \in S\) such that \(a^* = saa^*\) and \(a^* = a^*at\).
Theorem 14

Let $S$ be an involutive semigroup, let $a \in S$ be an MP-invertible element, let $D$ be the $\mathcal{D}$-class of $S$ containing $a$ and $a^*$. Let $a^* = uv$ be a trace factorization of $a^*$ with respect to an arbitrary $e \in D^\cdot$. Then $vau \in H_e$ and

$$a^+ = u(vau)^\# v.$$
Applications

Generalized inverses of complex matrices


  computation based on representations in terms of linear equations
  and the well-known method for solving matrix equations of the form $AXB = D$

existence criteria – given in terms of ranks (e.g., $\text{rank}(CAB) = \text{rank}(B) = \text{rank}(C)$)

Generalized inverses of fuzzy matrices


  computation based on our methods for solving equations and inequalities defined by residuated functions

Generalized inverses in residuated semigroups and quantales


a similar methodology