Identities in upper triangular tropical matrix semigroups and the bicyclic monoid

Marianne Johnson (University of Manchester)
Joint work with Laure Daviaud and Mark Kambites

AAA94 + NSAC 2017, 17th June 2017

Semigroup identities for matrices?

Let Σ be an alphabet, and consider two words $w, v \in \Sigma^+$.
Say that $w = v$ is a **semigroup identity** for S if $\varphi(w) = \varphi(v)$ for all semigroup morphisms $\varphi : \Sigma^+ \to S$.

Does the semigroup $M_n(X)$ satisfy an identity?

Semigroup identities for matrices?

Let Σ be an alphabet, and consider two words $w, v \in \Sigma^+$.
Say that $w = v$ is a **semigroup identity** for S if $\varphi(w) = \varphi(v)$ for all semigroup morphisms $\varphi : \Sigma^+ \to S$.

Does the semigroup $M_n(X)$ satisfy an identity?

Clearly depends upon X!

Semigroup identities for matrices?

Let Σ be an alphabet, and consider two words $w, v \in \Sigma^+$.
Say that $w = v$ is a **semigroup identity** for S if $\varphi(w) = \varphi(v)$ for all semigroup morphisms $\varphi : \Sigma^+ \to S$.

Does the semigroup $M_n(X)$ satisfy an identity?

Clearly depends upon X!

- There can be LOTS of identities (i.e. no finite basis)...
  M. Sapir/Volkov, 1980s:
  $M_n(K)$ for $n > 1$, K a finite field is not finitely based.

Semigroup identities for matrices?

Let Σ be an alphabet, and consider two words $w, v \in \Sigma^+$.
Say that $w = v$ is a **semigroup identity** for S if $\varphi(w) = \varphi(v)$ for all semigroup morphisms $\varphi : \Sigma^+ \to S$.

Does the semigroup $M_n(X)$ satisfy an identity?

Clearly depends upon X!

- There can be LOTS of identities (i.e. no finite basis)...
  M. Sapir/Volkov, 1980s:
  $M_n(K)$ for $n > 1$, K a finite field is not finitely based.

- Or none at all...
  Golubchik and Mikhalev, 1978:
  $M_n(K)$ for $n > 1$, K an infinite field does not satisfy a non-trivial semigroup identity.

The tropical semifield

Let $\mathbb{T}$ denote the **tropical semifield** $(\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$, where $\oplus$ and $\otimes$ denote two binary operations defined by:

$$x \oplus y := \max(x, y), \quad x \otimes y := x + y.$$

- Addition is idempotent. $x \oplus x = x.$
- 0 is 'one': $x \otimes 0 = 0 \otimes x = x.$
- $-\infty$ is 'zero': $x \oplus -\infty = -\infty \oplus x = x,$
  $x \otimes -\infty = -\infty \otimes x = -\infty$

The tropical semifield

Let $\mathbb{T}$ denote the **tropical semifield** $(\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$, where $\oplus$ and $\otimes$ denote two binary operations defined by:

$$x \oplus y := \max(x, y), \quad x \otimes y := x + y.$$

- Addition is idempotent. $\quad x \oplus x = x.$
- 0 is 'one': $\quad x \otimes 0 = 0 \otimes x = x.$
- $-\infty$ is 'zero': $\quad x \oplus -\infty = -\infty \oplus x = x,$ $\quad x \otimes -\infty = -\infty \otimes x = -\infty$

The **semigroup** $M_n(\mathbb{T})$ is the set of all $n \times n$ **tropical matrices**, with multiplication $\otimes$ defined in the obvious way:

$$\begin{pmatrix} 2 & 1 \\ 0 & 19 \end{pmatrix} \otimes \begin{pmatrix} -1 & -1 \\ -\infty & 4 \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ -1 & 23 \end{pmatrix}$$

The tropical semifield

Let $\mathbb{T}$ denote the **tropical semifield** $(\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$, where $\oplus$ and $\otimes$ denote two binary operations defined by:

$$x \oplus y := \max(x, y), \quad x \otimes y := x + y.$$

- Addition is idempotent. $x \oplus x = x.$
- 0 is 'one': $x \otimes 0 = 0 \otimes x = x.$
- $-\infty$ is 'zero': $x \oplus -\infty = -\infty \oplus x = x,$ $x \otimes -\infty = -\infty \otimes x = -\infty$

The **semigroup** $M_n(\mathbb{T})$ is the set of all $n \times n$ **tropical matrices**, with multiplication $\otimes$ defined in the obvious way:

$$\begin{pmatrix} 2 & 1 \\ 0 & 19 \end{pmatrix} \otimes \begin{pmatrix} -1 & -1 \\ -\infty & 4 \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ -1 & 23 \end{pmatrix}$$

We write $UT_n(\mathbb{T})$ to denote the subsemigroup of all upper triangular tropical matrices.

Tropical matrix identities

Does $M_n(\mathbb{T})$ satisfy a non-trivial semigroup identity?

- Yes, for $n = 2$, (Izhakian and Margolis, 2010).
- Yes, for $n = 3$ (Shitov, 2014).
- Open for $n > 3$.

Tropical matrix identities

Does $M_n(\mathbb{T})$ satisfy a non-trivial semigroup identity?

- Yes, for $n = 2$, (Izhakian and Margolis, 2010).
- Yes, for $n = 3$ (Shitov, 2014).
- Open for $n > 3$.

Finitely generated subsemigroups of $M_n(\mathbb{T})$ have polynomial growth (d'Alesandro and Pasku, 2003).

Tropical matrix identities

Does $M_n(\mathbb{T})$ satisfy a non-trivial semigroup identity?

- Yes, for $n = 2$, (Izhakian and Margolis, 2010).
- Yes, for $n = 3$ (Shitov, 2014).
- Open for $n > 3$.

Finitely generated subsemigroups of $M_n(\mathbb{T})$ have polynomial growth (d'Alesandro and Pasku, 2003).

Why do we care?

Tropical matrices can provide a tool for studying other interesting semigroups.

- Bicyclic monoid embeds into $UT_2(\mathbb{T})$ (Izhakian and Margolis, 2010).

Tropical matrix identities

Does $M_n(\mathbb{T})$ satisfy a non-trivial semigroup identity?

- Yes, for $n = 2$, (Izhakian and Margolis, 2010).
- Yes, for $n = 3$ (Shitov, 2014).
- Open for $n > 3$.

Finitely generated subsemigroups of $M_n(\mathbb{T})$ have polynomial growth (d'Alesandro and Pasku, 2003).

Why do we care?

Tropical matrices can provide a tool for studying other interesting semigroups.

- Bicyclic monoid embeds into $UT_2(\mathbb{T})$ (Izhakian and Margolis, 2010).
- Plactic monoid of rank 3 embeds into a direct product of copies of $UT_3(\mathbb{T})$ (Izhakian, 2017; Cain, Klein, Kubat, Malheiro, Okniński, 2017).

Tropical matrix identities

Does $M_n(\mathbb{T})$ satisfy a non-trivial semigroup identity?

- Yes, for $n = 2$, (Izhakian and Margolis, 2010).
- Yes, for $n = 3$ (Shitov, 2014).
- Open for $n > 3$.

Finitely generated subsemigroups of $M_n(\mathbb{T})$ have polynomial growth (d'Alesandro and Pasku, 2003).

Why do we care?

Tropical matrices can provide a tool for studying other interesting semigroups.

- Bicyclic monoid embeds into $UT_2(\mathbb{T})$ (Izhakian and Margolis, 2010).
- Plactic monoid of rank 3 embeds into a direct product of copies of $UT_3(\mathbb{T})$ (Izhakian, 2017; Cain, Klein, Kubat, Malheiro, Okniński, 2017).
- Free monogenic inverse monoid embeds into $UT_3(\mathbb{T})$.

Upper triangular tropical matrix identities

Does $UT_n(\mathbb{T})$ satisfy a non-trivial semigroup identity?

Upper triangular tropical matrix identities

Does $UT_n(\mathbb{T})$ satisfy a non-trivial semigroup identity?

- Yes! A couple of different constructions known. (Izhakian 2014, Okniński 2015, Taylor 2016)

Upper triangular tropical matrix identities

Does $UT_n(\mathbb{T})$ satisfy a non-trivial semigroup identity?

- Yes! A couple of different constructions known.
  (Izhakian 2014, Okniński 2015, Taylor 2016)

How can we tell if $w = v$ is an identity for $UT_n(\mathbb{T})$?

Upper triangular tropical matrix identities

Does $UT_n(\mathbb{T})$ satisfy a non-trivial semigroup identity?

- Yes! A couple of different constructions known. (Izhakian 2014, Okniński 2015, Taylor 2016)

How can we tell if $w = v$ is an identity for $UT_n(\mathbb{T})$?

- We provide a necessary and sufficient condition (in terms of certain tropical polynomials) for an identity $w = v$ to hold in $UT_n(\mathbb{T})$.

Upper triangular tropical matrix identities

Does $UT_n(\mathbb{T})$ satisfy a non-trivial semigroup identity?

- Yes! A couple of different constructions known. (Izhakian 2014, Okniński 2015, Taylor 2016)

How can we tell if $w = v$ is an identity for $UT_n(\mathbb{T})$?

- We provide a necessary and sufficient condition (in terms of certain tropical polynomials) for an identity $w = v$ to hold in $UT_n(\mathbb{T})$.
- This yields a polynomial time algorithm to verify whether a given identity is satisfied by $UT_n(\mathbb{T})$.

Upper triangular tropical matrix identities

Does $UT_n(\mathbb{T})$ satisfy a non-trivial semigroup identity?

- Yes! A couple of different constructions known. (Izhakian 2014, Okniński 2015, Taylor 2016)

How can we tell if $w = v$ is an identity for $UT_n(\mathbb{T})$?

- We provide a necessary and sufficient condition (in terms of certain tropical polynomials) for an identity $w = v$ to hold in $UT_n(\mathbb{T})$.
- This yields a polynomial time algorithm to verify whether a given identity is satisfied by $UT_n(\mathbb{T})$.

What is the variety generated by $UT_n(\mathbb{T})$?

Upper triangular tropical matrix identities

Does $UT_n(\mathbb{T})$ satisfy a non-trivial semigroup identity?

- Yes! A couple of different constructions known. (Izhakian 2014, Okniński 2015, Taylor 2016)

How can we tell if $w = v$ is an identity for $UT_n(\mathbb{T})$?

- We provide a necessary and sufficient condition (in terms of certain tropical polynomials) for an identity $w = v$ to hold in $UT_n(\mathbb{T})$.
- This yields a polynomial time algorithm to verify whether a given identity is satisfied by $UT_n(\mathbb{T})$.

What is the variety generated by $UT_n(\mathbb{T})$?

- For $n = 2$ we show that $UT_2(\mathbb{T})$ generates the same semigroup variety as the bicyclic monoid.

Upper triangular tropical matrix identities

Does $UT_n(\mathbb{T})$ satisfy a non-trivial semigroup identity?

- Yes! A couple of different constructions known. (Izhakian 2014, Okniński 2015, Taylor 2016)

How can we tell if $w = v$ is an identity for $UT_n(\mathbb{T})$?

- We provide a necessary and sufficient condition (in terms of certain tropical polynomials) for an identity $w = v$ to hold in $UT_n(\mathbb{T})$.
- This yields a polynomial time algorithm to verify whether a given identity is satisfied by $UT_n(\mathbb{T})$.

What is the variety generated by $UT_n(\mathbb{T})$?

- For $n = 2$ we show that $UT_2(\mathbb{T})$ generates the same semigroup variety as the bicyclic monoid.
- Several variants of these two also generate the same variety.

The case $n = 2$: Tropical polynomials

For $w \in \Sigma^*$, $s \in \Sigma$ define a formal tropical polynomial in variables x_t for $t \in \Sigma$ by:

$$f_s^w = \bigoplus_{w_i = s} \bigotimes_{t \in \Sigma} x_t^{\otimes \lambda_t^w(i-1)},$$

where $\lambda_t^w(k) = \# t$'s in the first k letters of w.

The case $n = 2$: Tropical polynomials

For $w \in \Sigma^*$, $s \in \Sigma$ define a formal tropical polynomial in variables x_t for $t \in \Sigma$ by:

$$f_s^w = \bigoplus_{w_i = s} \bigotimes_{t \in \Sigma} x_t^{\otimes \lambda_t^w(i-1)},$$

where $\lambda_t^w(k) = \#t$'s in the first k letters of w.

Theorem: Let $w, v \in \Sigma^+$.
The identity $w = v$ holds on $UT_2(\mathbb{T})$ if and only if $f_s^w(\underline{x}) = f_s^v(\underline{x})$ for all $s \in \Sigma$ and all $\underline{x} \in \mathbb{R}^\Sigma$.

The case $n = 2$: Two letter identities

In the case where w and v are words on a two letter alphabet, it suffices to compare four pairs of tropical polynomials in just **one** variable:

Corollary: Let $w, v \in \Sigma^+$, where $\Sigma = \{a, b\}$.
The identity $w = v$ holds on $UT_2(\mathbb{T})$ if and only if for all $x \in \mathbb{R}$ we have:

- $f_a^w(x, 1) = f_a^v(x, 1)$,
- $f_a^w(x, -1) = f_a^v(x, -1)$,
- $f_b^w(x, 1) = f_b^v(x, 1)$, and
- $f_b^w(x, -1) = f_b^v(x, -1)$.

The case $n = 2$: Example

$$f_s^u = \bigoplus_{u_i = s} \bigotimes_{t \in \Sigma} x_t^{\otimes \lambda_t^u (i-1)}$$

Let $w := abba\ ab\ abba$ and $v = abba\ ba\ abba$.

Then for example,

$$f_a^w(x_a, x_b) \;\; = \;\; x_a^0 \otimes x_b^0 \oplus x_a^1 \otimes x_b^2 \oplus x_a^2 \otimes x_b^2 \oplus x_a^3 \oplus x_b^3 \oplus x_a^4 \otimes x_b^5$$

$$f_a^v(x_a, x_b) \;\; = \;\; x_a^0 \otimes x_b^0 \oplus x_a^1 \otimes x_b^2 \oplus x_a^2 \otimes x_b^3 \oplus x_a^3 \otimes x_b^3 \oplus x_a^4 \otimes x_b^5$$

The case $n = 2$: Example

$$f_s^u = \bigoplus_{u_i = s} \bigotimes_{t \in \Sigma} x_t^{\otimes \lambda_t^u(i-1)}$$

Let $w := abba\ ab\ abba$ and $v = abba\ ba\ abba$.

Then for example,

$$f_a^w(x_a, x_b) = x_a^0 \otimes x_b^0 \oplus x_a^1 \otimes x_b^2 \oplus x_a^2 \otimes x_b^2 \oplus x_a^3 \oplus x_b^3 \oplus x_a^4 \otimes x_b^5$$

$$f_a^w(x, 1) = \max(0, x+2, 2x+2, 3x+3, 4x+5)$$

$$f_a^v(x_a, x_b) = x_a^0 \otimes x_b^0 \oplus x_a^1 \otimes x_b^2 \oplus x_a^2 \otimes x_b^3 \oplus x_a^3 \otimes x_b^3 \oplus x_a^4 \otimes x_b^5$$

$$f_a^v(x, 1) = \max(0, x+2, 2x+3, 3x+3, 4x+5)$$

The case $n = 2$: Example

$$f_s^u = \bigoplus_{u_i = s} \bigotimes_{t \in \Sigma} x_t^{\otimes \lambda_t^u(i-1)}$$

Let $w := abba\ ab\ abba$ and $v = abba\ ba\ abba$.

Then for example:

$$f_a^w(x_a, x_b) = x_a^0 \otimes x_b^0 \oplus x_a^1 \otimes x_b^2 \oplus x_a^2 \otimes x_b^2 \oplus x_a^3 \oplus x_b^3 \oplus x_a^4 \otimes x_b^5$$

$$f_a^w(x, 1) = \max(0, x+2, 2x+2, 3x+3, 4x+5)$$
$$\max(0, x+2, 3x+3, 4x+5)$$

$$f_a^v(x_a, x_b) = x_a^0 \otimes x_b^0 \oplus x_a^1 \otimes x_b^2 \oplus x_a^2 \otimes x_b^3 \oplus x_a^3 \otimes x_b^3 \oplus x_a^4 \otimes x_b^5$$

$$f_a^v(x, 1) = \max(0, x+2, 2x+3, 3x+3, 4x+5)$$
$$\max(0, x+2, 3x+3, 4x+5)$$

Connection to the bicyclic monoid

Izhakian and Margolis...

- Gave an embedding of the bicyclic monoid $\mathcal{B}$ into $UT_2(\mathbb{T})$.
  So, any identity satisfied by $UT_2(\mathbb{T})$ is also satisfied by $\mathcal{B}$.

- Proved that $UT_2(\mathbb{T})$ satisfies **Adjan's identity**:

  $$abba\ ab\ abba = abba\ ba\ abba,$$

  thus exhibiting a minimal length identity for $UT_2(\mathbb{T})$.

- Asked: Do $\mathcal{B}$ and $UT_2(\mathbb{T})$ generate the same variety?

Connection to the bicyclic monoid

Izhakian and Margolis...

- Gave an embedding of the bicyclic monoid $\mathcal{B}$ into $UT_2(\mathbb{T})$.
  So, any identity satisfied by $UT_2(\mathbb{T})$ is also satisfied by $\mathcal{B}$.
- Proved that $UT_2(\mathbb{T})$ satisfies **Adjan's identity**:

  $$abba\ ab\ abba = abba\ ba\ abba,$$

  thus exhibiting a minimal length identity for $UT_2(\mathbb{T})$.
- Asked: Do $\mathcal{B}$ and $UT_2(\mathbb{T})$ generate the same variety?

We use our tropical polynomials to answer this question...

Connection to the bicyclic monoid

Do $\mathcal{B}$ and $UT_2(\mathbb{T})$ generate the same variety?

Connection to the bicyclic monoid

Do $\mathcal{B}$ and $UT_2(\mathbb{T})$ generate the same variety?

Some observations about these two(?) varieties...

- Checking whether an identity holds amounts to checking certain linear inequalities.

  For $\mathcal{B}$: Pastijn, 2006. For $UT_2(\mathbb{T})$: DJK, 2017.

Connection to the bicyclic monoid

Do $\mathcal{B}$ and $UT_2(\mathbb{T})$ generate the same variety?

Some observations about these two(?) varieties...

- Checking whether an identity holds amounts to checking certain linear inequalities.
  For $\mathcal{B}$: Pastijn, 2006. For $UT_2(\mathbb{T})$: DJK, 2017.
- The variety generated is not finitely based.
  For $\mathcal{B}$: Shneerson, 1989. ($\mathcal{B}$ has infinite axiomatic rank.)
  For $UT_2(\mathbb{T})$: Chen, Hu, Luo, O. Sapir, 2016.

Connection to the bicyclic monoid

Do $\mathcal{B}$ and $UT_2(\mathbb{T})$ generate the same variety?

Some observations about these two(?) varieties...

- Checking whether an identity holds amounts to checking certain linear inequalities.
  For $\mathcal{B}$: Pastijn, 2006. For $UT_2(\mathbb{T})$: DJK, 2017.
- The variety generated is not finitely based.
  For $\mathcal{B}$: Shneerson, 1989. ($\mathcal{B}$ has infinite axiomatic rank.)
  For $UT_2(\mathbb{T})$: Chen, Hu, Luo, O. Sapir, 2016.

Theorem: $\mathcal{B}$ and $UT_2(\mathbb{T})$ generate the same variety.

Connection to the bicyclic monoid

Do $\mathcal{B}$ and $UT_2(\mathbb{T})$ generate the same variety?

Some observations about these two(?) varieties...

- Checking whether an identity holds amounts to checking certain linear inequalities.
  For $\mathcal{B}$: Pastijn, 2006. For $UT_2(\mathbb{T})$: DJK, 2017.
- The variety generated is not finitely based.
  For $\mathcal{B}$: Shneerson, 1989. ($\mathcal{B}$ has infinite axiomatic rank.)
  For $UT_2(\mathbb{T})$: Chen, Hu, Luo, O. Sapir, 2016.

Theorem: $\mathcal{B}$ and $UT_2(\mathbb{T})$ generate the same variety.

- We show that if $w \neq v$ in $UT_2(\mathbb{T})$, then there is a morphism from Σ^+ to the image of $\mathcal{B}$ in $UT_2(\mathbb{T})$ that falsifies the identity.

Variants

Let $\mathcal{T} = \mathbb{R}, \mathbb{Q}, \mathbb{Z}$. Can also consider: $\mathcal{T}_{\max} := (\mathcal{T} \cup \{-\infty\}, \oplus, \otimes)$.

- Define the semigroup $\mathcal{B}_{\mathcal{T}} := \mathcal{T} \times \mathcal{T}$ via the product

$$(a, b) \cdot (c, d) = (a - b + \max(b, c), \quad d - c + \max(b, c)).$$

- For each $\mathcal{T}$ as above we have

$$\mathcal{B} \cong \mathcal{B}_{\mathbb{N}_0} \subseteq \mathcal{B}_{\mathcal{T}} \hookrightarrow UT_2(\mathcal{T}_{\max}) \hookrightarrow UT_2(\mathbb{T}).$$

- Since $\mathcal{B}$ and $UT_2(\mathbb{T})$ generate the same semigroup variety, it follows that each of the intermediate variants above must satisfy exactly the same semigroup identities as these two.

Generalisation

Given a set Γ equipped with a reflexive and transitive relation R, define a semigroup:

$\Gamma(\mathbb{T}) = \{A \in \mathbb{T}^{\Gamma \times \Gamma} : A \text{ is bounded above and } A_{i,j} \neq -\infty \Rightarrow iRj\}$,

with product

$$(A \otimes B)(i,j) = \bigoplus_{k \in \Gamma} A(i,k) \otimes B(k,j).$$

Examples to keep in mind...

- $\Gamma = \{1, \ldots, n\}$, R the complete relation: $M_n(\mathbb{T})$.
- $\Gamma = \{1, \ldots, n\}$, R the natural order: $UT_n(\mathbb{T})$.

- If Γ is a poset in with a global bound on the length of chains, then checking whether an identity holds in $\Gamma(\mathbb{T})$ is easy...

Bounded chains and tropical polynomials

Theorem: Γ a poset with maximum chain length n.

$w = v$ holds in $\Gamma(\mathbb{T})$ if and only if $f^w_{u,\rho}(\underline{x}) = f^v_{u,\rho}(\underline{x})$ for all:

- words $u \in \Sigma^*$ of length at most $n-1$;
- chains $\rho = (\rho_0 \preceq \rho_1 \preceq \cdots \preceq \rho_{|u|})$ in Γ; and
- values $\underline{x} \in \mathbb{R}^{\Sigma \times \Gamma}$.

Formal tropical polynomials in variables $x(s,i)$, $(s \in \Sigma, i \in \Gamma)$:

$$f^w_{u,\rho} := \bigoplus \bigotimes_{s \in \Sigma} \bigotimes_{k=0}^{|u|} x(s,\rho_k)^{\beta^w_s(\alpha_k,\alpha_{k+1})},$$

where the sum ranges over all sequences:
$0 = \alpha_0 < \alpha_1 < \cdots < \alpha_{|u|} < \alpha_{|u|+1} = |w|+1$ in which $w_{\alpha_k} = u_k$,
and where $\beta^w_s(\alpha_k,\alpha_{k+1}) = \sharp s$ strictly between w_{α_k} and $w_{\alpha_{k+1}}$.

Bounded chains and tropical polynomials

Corollary: Γ a poset with maximum chain length n.

- $\Gamma(\mathbb{T})$ and $UT_n(\mathbb{T})$ generate the same variety.
- $\Gamma(\mathbb{T})$ satisfies a non-trivial semigroup identity.

- The polynomials that must agree for $w = v$ to hold in $UT_n(\mathbb{T})$ are, up to a change of variables, the same as those that must agree for $w = v$ to hold in $\Gamma(\mathbb{T})$.
- Since a non-trivial identity holds in each $UT_n(\mathbb{T})$, the result follows.

Example: The free monogenic inverse monoid

$$\mathcal{I} \;\cong\; \{(i,j,k) \in \mathbb{Z}^3 : i,j \geqslant 0, -j \leq k \leq i\}.$$

with product

$$(i,j,k)\cdot(i',j',k') = (\max(i,\ i'+k),\ \ \max(j,\ j'-k),\ \ k+k')$$

- Taking $\Gamma = \{1,2,3\}$ with partial order $1,2 \preceq 3$ can see that $\mathcal{I} \hookrightarrow \Gamma(\mathbb{T}) \subseteq UT_3(\mathbb{T})$

$$(i,j,k) \mapsto \begin{pmatrix} k & -\infty & i \\ -\infty & -k & j \\ -\infty & -\infty & 0 \end{pmatrix}.$$

- Follows that $\mathcal{I}$ satisfies all identities satisfied by $UT_2(\mathbb{T})$ (and hence $\mathcal{B}$).
- Can show that $\mathcal{I}$ does not embed in $UT_2(\mathbb{T})$.

Further connections to the bicyclic monoid?

Izhakian and Margolis, 2010: Proved that...

- $UT_2(\mathbb{T})$ satisfies **Adjan's identity**:

$$abba\ ab\ abba = abba\ ba\ abba.$$

- $M_2(\mathbb{T})$ satisfies the identity:

$$a^2b^2b^2a^2 \quad a^2b^2 \quad a^2b^2b^2a^2 = a^2b^2b^2a^2 \quad b^2a^2 \quad a^2b^2b^2a^2.$$

Does "squaring" an identity for $UT_2(\mathbb{T})$ always yield an identity for $M_2(\mathbb{T})$?

No! E.g. $ab^2\ ab\ ab\ a^2b = ab^2\ a^2\ b^2\ a^2b$ holds in $UT_2(\mathbb{T})$.

But $a^2b^4\ a^2b^2\ a^2b^2\ a^4b^2 = a^2b^4\ a^4\ b^4\ a^4b^2$ doesn't hold in $M_2(\mathbb{T})$.

Tropical matrix identities are long

Izhakian and Margolis, 2010: $M_2(\mathbb{T})$ satisfies the identity:

$$a^2b^2b^2a^2 \quad a^2b^2 \quad a^2b^2b^2a^2 = a^2b^2b^2a^2 \quad b^2a^2 \quad a^2b^2b^2a^2.$$

Does a shorter identity hold for $M_2(\mathbb{T})$?

DJ, 2017: Yes! Although not much shorter...

- List of necessary conditions for identities over a two letter alphabet; rules out all words of length < 17.
- Explicit calculation yields

$$a^2b^3\ a^3\ ba\ ba\ b^3a^2 = a^2b^3\ ab\ ab\ a^3\ b^3a^2.$$

The case $n = 2$: How the polynomials arise

Consider morphisms $\varphi : \Sigma^+ \to UT_2(\mathbb{T})$ of the form:

$$(\bigstar) \quad \varphi(s) = \begin{pmatrix} x_s & x'_s \\ -\infty & 0 \end{pmatrix}, \text{ for all } s \in \Sigma.$$

The case $n = 2$: How the polynomials arise

Consider morphisms $\varphi : \Sigma^+ \to UT_2(\mathbb{T})$ of the form:

$$(\bigstar) \quad \varphi(s) = \begin{pmatrix} x_s & x'_s \\ -\infty & 0 \end{pmatrix}, \text{ for all } s \in \Sigma.$$

- For all $w \in \Sigma^+$, can see that:

$$\varphi(w)_{1,1} \quad = \quad \sum_{t \in \Sigma} |w|_t x_t, \quad \varphi(w)_{2,1} = -\infty, \quad \varphi(w)_{2,2} = 0$$

The case $n = 2$: How the polynomials arise

Consider morphisms $\varphi : \Sigma^+ \to UT_2(\mathbb{T})$ of the form:

$$(\bigstar) \quad \varphi(s) = \begin{pmatrix} x_s & x'_s \\ -\infty & 0 \end{pmatrix}, \text{ for all } s \in \Sigma.$$

- For all $w \in \Sigma^+$, can see that:

$$\varphi(w)_{1,1} = \sum_{t \in \Sigma} |w|_t x_t, \quad \varphi(w)_{2,1} = -\infty, \quad \varphi(w)_{2,2} = 0$$

$$\varphi(w)_{1,2} = \max_{s \in \Sigma} \left(\max_{w=usv} \left(\sum_{t \in \Sigma} |u|_t x_t + x'_s + 0 \right) \right)$$

The case $n = 2$: How the polynomials arise

Consider morphisms $\varphi : \Sigma^+ \to UT_2(\mathbb{T})$ of the form:

$$(\bigstar) \quad \varphi(s) = \begin{pmatrix} x_s & x'_s \\ -\infty & 0 \end{pmatrix}, \text{ for all } s \in \Sigma.$$

- For all $w \in \Sigma^+$, can see that:

$$\varphi(w)_{1,1} = \sum_{t\in\Sigma} |w|_t x_t, \quad \varphi(w)_{2,1} = -\infty, \quad \varphi(w)_{2,2} = 0$$

$$\begin{aligned} \varphi(w)_{1,2} &= \max_{s\in\Sigma}\left(\max_{w=usv}\left(\sum_{t\in\Sigma} |u|_t x_t + x'_s + 0\right)\right) \\ &= \max_{s\in\Sigma}\left(x'_s + \max_{w_i=s}\left(\sum_{t\in\Sigma} \lambda_t^w(i-1)x_t\right)\right) \end{aligned}$$

where $|w|_t$ = the number of occurrences of the letter t in w, and $\lambda_t^w(k) = \#t$'s in the first k letters of w.