Decent Mal'cev conditions which hold in all locally finite congruence meet-semidistributive varieties I

Petar Marković
joint work with Nemanja Draganić, Vlado Uljarević and Samir Zahirović

University of Novi Sad, Serbia

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A strong Mal'cev condition is just a finite set of identities Σ. Σ is *realized* in a variety $\mathcal{V}$ (or algebra $\mathbf{A}$) if there exists a way to interpret its operations as *terms* of $\mathcal{V}$ (or of $\mathbf{A}$) so that the identities of Σ become true in $\mathcal{V}$ (or $\mathbf{A}$) after this "interpretation".

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A Mal'cev condition is a directed family of strong Mal'cev conditions, i.e. for any pair there is a more general (weaker) one in the family. Then there must be one of them which is realized.

Strong Mal'cev properties and examples

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4. Realization of some idempotent strong Mal'cev condition which fails in the variety $\mathcal{S}ets$ (Olšák, 2016).

More examples

Examples (Kozik, Krokhin, Valeriote and Willard, 2015)

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Problem

Is congruence meet-semidistributivity a strong Mal'cev property of all varieties?

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Let $\mathbb{A}$ be a finite relational structure. CSP($\mathbb{A}$) has bounded width iff $(A; Pol(\mathbb{A}))$ generates a congruence meet-semidistributive variety.

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Let k and l be fixed. For any $\mathbb{A}$ and any instance of $CSP(\mathbb{A})$ an equivalent (k, l)-minimal instance can be constructed in polynomial time.

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Theorem (Barto, 2016)

$CSP(\mathbb{A})$ has bounded width iff it has width $(2, 3)$.

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3. $\mathcal{V}$ *realizes (idempotence and)* $f(x,x,y) \approx f(x,y,y)$ *and* $f(x,y,x) \approx g(x,x,y) \approx g(x,y,x) \approx g(y,x,x)$ *(JMMM, 2016)*

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All three strong Mal'cev characterizations by applying that $(2,3)$-minimal nonempty CSP instances compatible with $\mathbf{F}_{\mathcal{V}}(x,y)$ have a solution. The first one by the Pigeonhole principle, the other two by Ramsey proofs.

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Theorem (Brady, 2016 / DMUZ, 2017)

Let $\mathcal{V}$ be a locally finite congruence meet-semidistributive variety. There exists a binary $\mathcal{V}$-term $u(x, y)$ such that for any $X = \{x_1, \dots, x_k\}$ and maximal intersecting family $\mathcal{F}$ on X, there exists an idempotent $\mathcal{V}$-term $t_{\mathcal{F}}(x_1, \dots, x_k)$, such that $\mathcal{V} \models t_{\mathcal{F}}(x_1^U, \dots, x_k^U) \approx u(x, y)$ for each $U \in \mathcal{F}$.

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Corollary (Brady, 2016)

A locally finite variety is congruence meet-semidistributive iff there exists an idempotent $\mathcal{V}$-term $f(x, y, z)$ such that $\mathcal{V} \models f(x, x, y) \approx f(x, y, x) \approx f(y, x, x) \approx f(f(x, x, y), f(x, x, y), f(x, y, y))$.

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The condition holds in all locally finite SD(∧) varieties iff no class of the extended equivalence relation contains a pair of disjoint subsets and also a pair of subsets whose union is everything. Poly-time checkable!

Main theorem

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Let Σ be a linear, idempotent strong Mal'cev condition in which only two variables occur. TFAE:

1. *Σ is realized in $\mathbf{D}$, but not in any nontrivial finite module.*
2. *For any locally finite variety $\mathcal{V}$, $\mathcal{V}$ realizes Σ iff $\mathcal{V}$ is congruence meet-semidistributive.*

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Here disjointness on a poset means a symmetric binary relation such that if $a||b$ then

$a{\downarrow} \, || \, b{\downarrow}$, and

for all $a' \leq a$ and $b' \leq b$, $a'||b'$.

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Imitate Olšák? Possibly. Jovanović terms do appear in the shortest sequence of Willard terms so they may be a candidate (Brady, –, just fooling around a few days ago). Not known for longer Willard sequences. Here they are (idempotence +):

$$w(x,x,y) \approx w(x,y,x) \approx w(y,x,x) \approx p(x,y,x)$$

$$p(x,x,y) \approx p(x,y,y).$$

THANK YOU,

SINIŠA, BRANKO AND REINHARD

FOR ALL YOU HAVE DONE!