A Ramsey theorem for relational structures consisting of several partial orders

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Ramsey Theory

Finite Ramsey Theorem.
For all $a, b \in \mathbb{N}$ and $k \geqslant 2$ there is a $c \in \mathbb{N}$ such that $c \longrightarrow (b)^a_k$.

Here, $c \longrightarrow (b)^a_k$ means:

for every c-element set C and every coloring

$$\chi : \binom{C}{a} \to k$$

there is a b-element set $B \subseteq C$ such that $|\chi(\binom{B}{a})| = 1$.

Frank P. Ramsey
1903 – 1930
Image courtesy of Wikipedia

Structural Ramsey theory

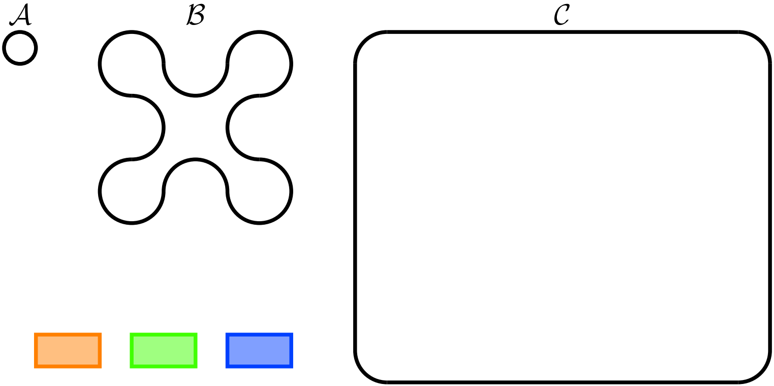
Deep structural property developed in the 1970's by Erdős, Graham, Leeb, Rothschild, Rödl, Nešetřil and many more.

Instead of sets, consider structures!

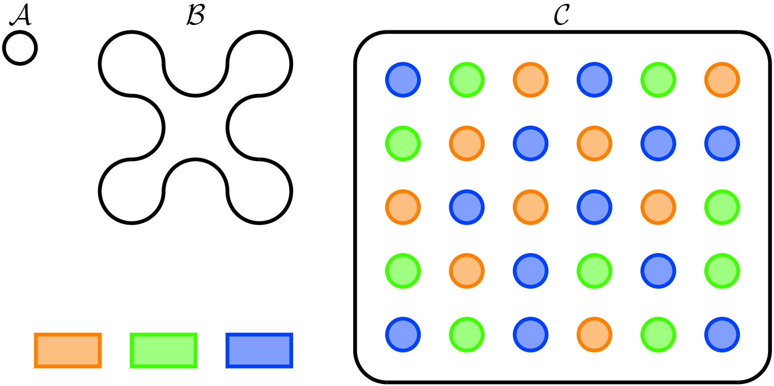
Definition. A class **K** of finite structures has the Ramsey property if:

> for all $\mathcal{A}, \mathcal{B} \in \mathbf{K}$ such that $\mathcal{A} \hookrightarrow \mathcal{B}$ and all $k \geqslant 2$ there is a $\mathcal{C} \in \mathbf{K}$ such that $\mathcal{C} \longrightarrow (\mathcal{B})^{\mathcal{A}}_k$.

$\mathcal{C} \longrightarrow (\mathcal{B})^{\mathcal{A}}_{k}$

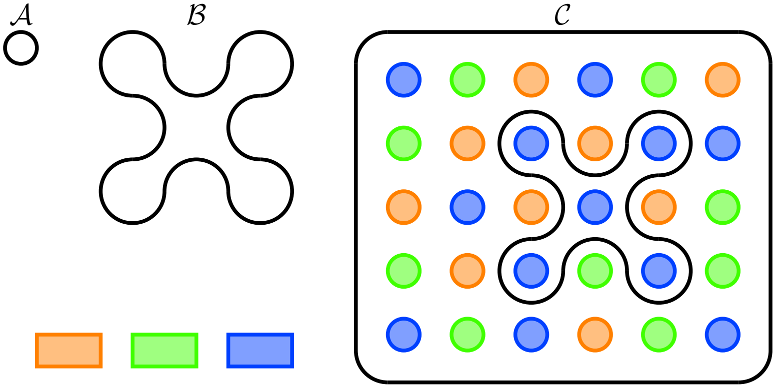


$\mathcal{C} \longrightarrow (\mathcal{B})^{\mathcal{A}}_k$



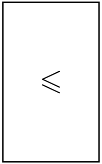
for every coloring $\chi : \binom{\mathcal{C}}{\mathcal{A}} \to k$

$\mathcal{C} \longrightarrow (\mathcal{B})^{\mathcal{A}}_k$



there is a $\tilde{\mathcal{B}} \in \binom{\mathcal{C}}{\mathcal{B}}$ such that $\left|\chi\left(\binom{\tilde{\mathcal{B}}}{\mathcal{A}}\right)\right| = 1$.

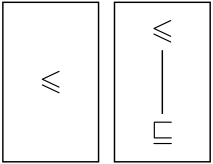
Structural Ramsey theory



Example.

- finite chains: $(A, \leqslant)$
  [Ramsey 1930]

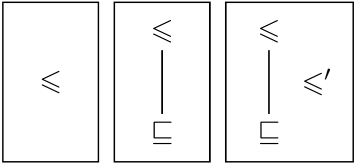
Structural Ramsey theory



Example.

- finite posets with a linear extension: $(A, \sqsubseteq, \leqslant)$ [Nešetřil, Rödl 1984]

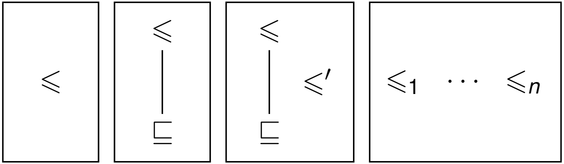
Structural Ramsey theory



Example.

- finite posets with a linear extension and another linear order: $(A, \sqsubseteq, \leqslant, \leqslant')$
  [Sokić 2010]

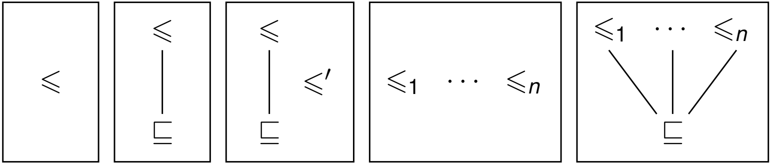
Structural Ramsey theory



Example.

- a set with n linear orders: $(A, \leqslant_1, \ldots, \leqslant_n)$
  [Sokić 2010]

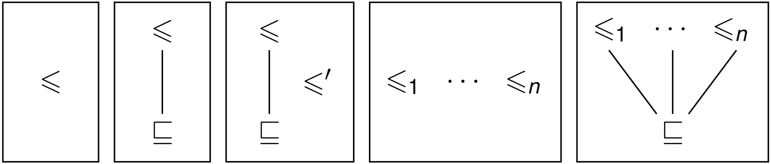
Structural Ramsey theory



Example.

- finite posets with n linear extensions: $(A, \sqsubseteq, \leqslant_1, \ldots, \leqslant_n)$ [Solecki, Zhao 2017]

Structural Ramsey theory



In each of these cases we have:

- a class of structures with several ordering relations,
- the relations are required to form a fixed partially ordered set under set inclusion, and
- the maximal elements in this poset of relations are linear orders.

Multiposets

$\mathcal{T} = (\{1, 2, \ldots, t\}, \preccurlyeq)$, $t \geqslant 1$ — a poset

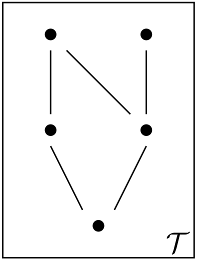
A $\mathcal{T}$-multiposet is a structure

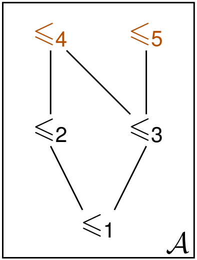
$$\mathcal{A} = (A, \leqslant_1, \ldots, \leqslant_t)$$

where:

- $\leqslant_1, \ldots, \leqslant_t$ are partial orders on A,
- if i is a maximal element of $\mathcal{T}$ then $\leqslant_i$ is a linear order, and
- if $i \preccurlyeq j$ in $\mathcal{T}$ then $(\leqslant_i) \subseteq (\leqslant_j)$.

K$(\mathcal{T})$ — all finite $\mathcal{T}$-multiposets

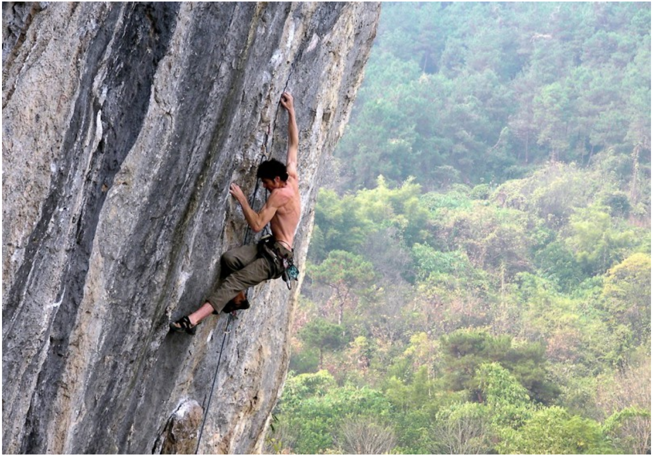




Multiposets

The "Multiposet Ramsey" Theorem. [Draganić, M. 2017+]
The class $\mathbf{K}(\mathcal{T})$ *has the Ramsey property for every template* $\mathcal{T}$*.*

Proving the Ramsey property



Proving the Ramsey property

The Ramsey property in a category

Let $\mathbb{C}$ be a category and $A, B, C \in \mathrm{Ob}(\mathbb{C})$.

$C \longrightarrow (B)^A_k$ if:

- $\hom(A, B) \neq \varnothing$, $\hom(B, C) \neq \varnothing$, and
- for every mapping $\chi : \hom(A, C) \to k$ there is a $\mathbb{C}$-morphism $w : B \to C$ such that $|\chi(w \cdot \hom(A, B))| = 1$.

A category $\mathbb{C}$ has the Ramsey property if:

for all $k \geqslant 2$ and all $A, B \in \mathrm{Ob}(\mathbb{C})$ such that $\hom(A, B) \neq \varnothing$ there is a $C \in \mathrm{Ob}(\mathbb{C})$ satisfying $C \longrightarrow (B)^A_k$

Transfer principles

- Products of categories.
- Categorical equivalence.
- Adjunctions.
- Pre-adjunctions.
- Passing to a subcategory.

Transfer principles

Example. The category $\mathbb{C}$ of finite posets with a linear extension (with embeddings) has the Ramsey property.

$\mathrm{GR}(A, X)$

$$k \xrightarrow[(w_i \in A \cup \{x_1, \ldots, x_k\})]{w = w_1 w_2 \ldots w_n} n$$

$$\begin{tikzcd} \mathrm{GR}(A,X) \arrow[d, shift left=3ex, "G"] \arrow[d, phantom, "\text{``}\dashv\text{''}"] \\ \mathbb{C} \arrow[u, shift left=3ex, "F"] \end{tikzcd}$$

$\mathbb{C}$

$$(A, \sqsubseteq, \leqslant) \hookrightarrow (B, \sqsubseteq', \leqslant')$$

Diagrams in a category

A binary diagram $F : \Delta \to \mathbb{C}$:

$$
\begin{tikzcd}
\bullet & \bullet & \bullet \\
\bullet \arrow[u] \arrow[ur] & \bullet \arrow[ul] \arrow[ur] & \bullet \arrow[ul] \arrow[u]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
B_1 & B_2 & B_3 \\
A_1 \arrow[u, "f_1"] \arrow[ur, "f_2"'] & A_2 \arrow[ul, "f_3"'] \arrow[ur, "f_4"] & A_3 \arrow[ul, "f_5"] \arrow[u, "f_6"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
\Delta \arrow[rrr, "F"] & & & \mathbb{C}
\end{tikzcd}
$$

Diagrams in a category

A consistent binary diagram $F : \Delta \to \mathbb{C}$:

$$
\begin{tikzcd}
\bullet & \bullet & \bullet \\
\bullet \arrow[u] \arrow[ur] & \bullet \arrow[ul] \arrow[ur] & \bullet \arrow[ul] \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \exists C & \\
B_1 \arrow[ur, dotted] & B_2 \arrow[u, dotted] & B_3 \arrow[ul, dotted] \\
A_1 \arrow[u, "f_1"] \arrow[ur, "f_2"'] & A_2 \arrow[ul, "f_3"'] \arrow[ur, "f_4"] & A_3 \arrow[ul, "f_5"] \arrow[u, "f_6"']
\end{tikzcd}
$$

$$\Delta \xrightarrow{\quad F \quad} \mathbb{C}$$

Diagrams in a category

A subcategory $\mathbb{D}$ of $\mathbb{C}$ is closed for binary diagrams if every binary diagram $F : \Delta \to \mathbb{D}$ which is consistent in $\mathbb{C}$ is also consistent in $\mathbb{D}$.

The "Closed Subcategory" Theorem. [M 2017+]
Let $\mathbb{C}$ be a category such that

- $\hom_{\mathbb{C}}(A, B)$ *is finite for all* $A, B \in \mathrm{Ob}(\mathbb{C})$, *and*
- *every morphism in $\mathbb{C}$ is monic.*

Let $\mathbb{D}$ be a (not necessarily full) subcategory of $\mathbb{C}$ such that

- *$\mathbb{D}$ is closed for binary diagrams.*

If $\mathbb{C}$ has the Ramsey property then $\mathbb{D}$ has the Ramsey property.

The proof of the "Multiposet Ramsey" Theorem

Step 1. The following classes have the Ramsey property:

- finite chains: $(A, \leqslant)$ [Ramsey 1930],
- finite posets with a linear extension: $(A, \sqsubseteq, \leqslant)$ [Nešetřil, Rödl 1984].

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Step 2. Starting from the these two classes of structures

- we build a “large” category $\mathbb{C}$ which has the Ramsey property, and s.t.
- $\mathbf{K}(\mathcal{T})$ is a subcategory of $\mathbb{C}$.

The proof of the “Multiposet Ramsey” Theorem

Step 1. The following classes have the Ramsey property:

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Step 2. Starting from the these two classes of structures

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- $\mathbf{K}(\mathcal{T})$ is a subcategory of $\mathbb{C}$.

Step 3.

- We show that $\mathbf{K}(\mathcal{T})$ is closed for binary diagrams in $\mathbb{C}$, and
- infer the Ramsey property for $\mathbf{K}(\mathcal{T})$ using the “Closed Subcategory” Theorem.

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