MORITA EQUIVALENCE OF SEMIGROUPS REVISITED: FIRM SEMIGROUPS

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Monoids

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Corollary 2.
The categories of all acts over two arbitrary semigroups are equivalent if and only if the two semigroups are isomorphic.

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S factorisable $\Longleftrightarrow$ μ_S surjective

S is called a **firm semigroup** if S_S (or ${}_SS$) is a firm act

Semigroups, acts 2

A_S *non-singular act*: if $as = a's$ for all $s \in S$, then $a = a'$ $(a, a' \in A)$

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$\mathrm{Fx} - \mathrm{Act}_S$ all fixed acts: those acts A_S for which the mapping

$$S \otimes S\mathrm{Act}(S, A) \to A : \ s \otimes f \mapsto (s)f$$

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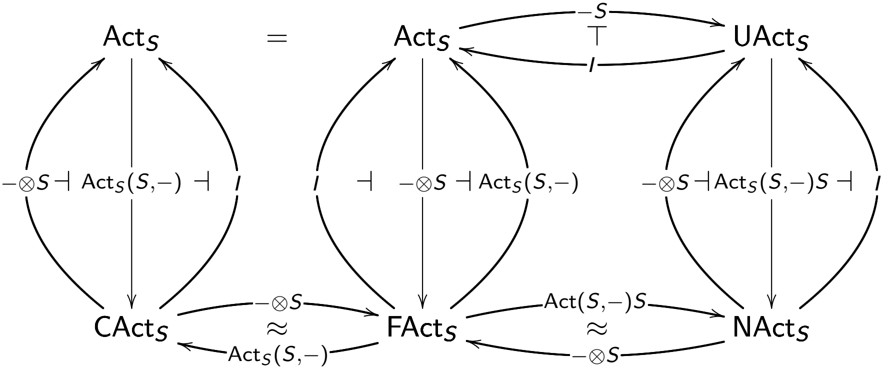
Adjoint situations, equivalences 1

Let S be a firm semigroup.



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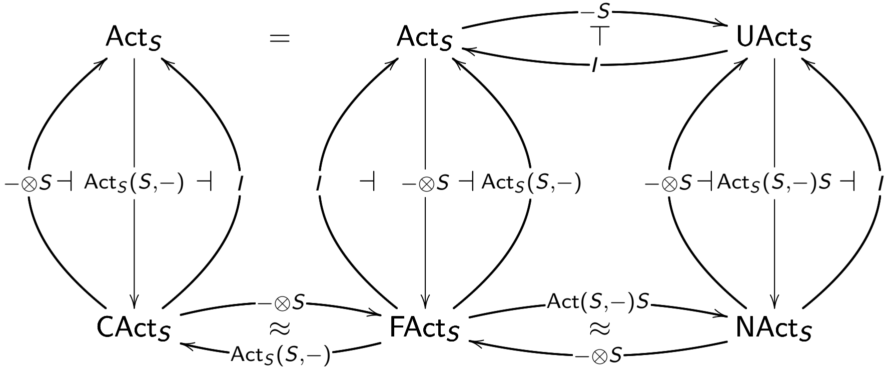

The equivalence between CAct_S and NAct_S can be given as

$$\mathsf{CAct}_S \underset{\mathsf{Act}_S(S,-)}{\overset{-S}{\rightleftarrows}} \mathsf{NAct}_S.$$

Adjoint situations, equivalences 2

FAct_S is an essential colocalization of UAct_S with coreflection $- \otimes S\mathsf{I}$;
NAct_S is an essential localization of UAct_S with reflection $\mathsf{Act}_S(S, -)S\mathsf{I}$.

Equivalence functors between categories of firm acts

Let S and T be firm semigroups and ${}_S P_T$ be a biact such that P_T is firm. Then the functor $- \otimes P \colon \mathsf{FAct}_S \longrightarrow \mathsf{FAct}_T$ is left adjoint to the functor $\mathsf{Act}_T(P, -) \otimes S \colon \mathsf{FAct}_T \longrightarrow \mathsf{FAct}_S$.

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Let S and T be firm semigroups and $F : \mathsf{FAct}_S \to \mathsf{FAct}_T$ and $G : \mathsf{FAct}_S \to \mathsf{FAct}_T$ be mutually inverse equivalence functors. Then

$$F \cong \mathsf{Act}_S(G(T), -) \otimes T \quad \text{a} \quad G \cong \mathsf{Act}_T(F(S), -) \otimes S.$$

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$$F \cong -\otimes F(S),$$
$$G \cong -\otimes G(T).$$

Moreover, the left acts ${}_SF(S)$ and ${}_TG(T)$ are firm.

Morita contexts

a sixtuple $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$, where S and T are semigroups, ${}_SP_T \in {}_S\mathrm{Act}_T$ ands ${}_TQ_S \in {}_T\mathrm{Act}_S$ are biacts, and

$$\theta : {}_S(P \otimes Q)_S \to {}_SS_S, \quad \phi : {}_T(Q \otimes P)_T \to {}_TT_T$$

are biact morphisms such that, for every $p, p' \in P$ and $q, q' \in Q$,

$$\theta(p \otimes q)p' = p\phi(q \otimes p'), \quad q\theta(p \otimes q') = \phi(q \otimes p)q'.$$

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A Morita context is

- *unitary* if ${}_SP_T$ and ${}_TQ_S$ are unitary biacts,
- *surjective* if θ and ϕ are surjective,
- *bijective* if θ and ϕ are bijective.

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Earlier results:

Lawson

(Right) Morita equivalence and strong Morita equivalence coincide for semigroups with local units.

Morita equivalence and strong Morita equivalence 2

Chen–Shum

For arbitrary factorisable semigroups S és T, the categories NAct_S and NAct_T are equivalent if and only if the semigroups S/ζ_S and T/ζ_T are strongly Morita equivalent, where the congruence ζ_A is defined, for an act A_S, by

$$\zeta_A = \{(a_1, a_2) \in A^2 \mid a_1 s = a_2 s \text{ for all } s \in S\}.$$

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Laan–Márki

Let S and T be fair semigroups such that $U(S)$ and $U(T)$ have common weak local units, where $U(S) = \{s \in S \mid s = us = sv \text{ for some } u, v \in S\}$ (this is an ideal in S). Then S and T are right Morita equivalent if and only if $U(S)$ and $U(T)$ are strongly Morita equivalent.

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