Solving weakly linear inequalities for matrices over max-plus semiring and applications to automata theory

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Semirings were first introduced by H.S. Vandiver in 1934, but implicitly they had appeared earlier in studies on the theory of ideals of rings and on the axiomatization of the natural numbers.

The theory of semirings has been developed in five principal directions:

- pure theoretical investigations;
- investigations related to theoretical arithmetic and number theory;
- investigations connected to logic in a broad sense, including non-classical and multi-valued logics,
- an algebraic approach to some geometrical and topological investigations and to the theory of differential equations;
- an algebraic tool for some investigations in the theory of automata, in the theory of formal languages, optimization theory and other branches of of applied mathematics.

Nowadays they have both a well elaborated algebraic theory, as well as important practical applications.

- K. Glazek, A Guide to the Literature on Semirings and their Applications in Mathematics and Information Science with Complete Bibliography, Springer-Science+Buissines Media, B.V., 2002.

Among the most studied and applied types of semirings are those with idempotent addition, called additively idempotent semirings. Applications in many areas of mathematics, computer science and operation research:

- in the theory of automata and formal languages,
- optimization theory,
- idempotent analysis,
- theory of programming languages,
- data analysis,
- discrete event systems theory,
- algebraic modeling of fuzziness and uncertainty,
- algebra of formal processes, etc.

An algebraic structure $\mathbf{S} = (S, \oplus, \otimes, 0, 1)$ is called a semiring if:

(S1) $(S, \oplus, 0)$ is a commutative monoid,

(S2) $(S, \otimes, 1)$ is a monoid,

(S3) multiplication distributes over addition:

$$(x \oplus y) \otimes z = x \otimes z \oplus y \otimes z,$$
$$z \otimes (x \oplus y) = z \otimes x \oplus z \otimes y, \qquad x, y, z \in S,$$

(S4) 0 is absorbing element:

$$0 \otimes x = x \otimes 0 = 0, \qquad x \in S.$$

(S5) $\mathbf{S}$ is additively idempotent semiring if

$$a \oplus a = a, \quad a \in S.$$

Any additively idempotent semiring is naturally ordered by

$$a \leqslant b \iff a \oplus b = b.$$

Every element of an additively idempotent semiring is non-negative, i.e.

$$0 \leqslant a.$$

- If a semiring fails to be a ring, it is by the absence of additive inverses.
- Using the natural ordering, it is possible to define in a usual way the notions of upper and lower bounds, bounded sets and $\bigvee M$ and $\bigwedge N$, etc.

J. S. Golan, Semirings and their Applications, Kluwer Academic Publisher, 1999.

Max-plus semiring is the semiring which the set $\mathbb{R}_{max} = \{-\infty\} \cup \mathbb{R}$ forms with operations $\oplus$ and $\otimes$ defined by

$$a \oplus b = \max(a, b) \quad \text{and} \quad a \otimes b = a + b, \quad \text{for all } a, b \in \mathbb{R}_{max}.$$

The zero element is $\varepsilon = -\infty$, i.e.,

$$a \oplus \varepsilon = \varepsilon \oplus a = a, \quad \text{for every } a \in \mathbb{R}_{max},$$

The unit element is $\epsilon = 0$, i.e.,

$$a \otimes \epsilon = \epsilon \otimes a = a, \quad \text{for every } a \in \mathbb{R}_{max}.$$

The natural order endows $\mathbb{R}_{max}$ with a sup-semilattice structure for which $a \oplus b$ is the least upper bound of the set $\{a, b\}$.

Residuated semirings

- A right residual $a\backslash b$ of b by a is the greatest solution (if it exists) of $a \cdot x \leqslant b$.
- A left residual b/a of b by a is the greatest solution (if it exists) of $x \cdot a \leqslant b$.

- Such residuals need not always exist, but if they exist they are unique;
- If the semiring is commutative, we need not to make distinction between right and left residuals;
- We say that the idempotent semiring S is complete if any family has a supremum, and if the product distributes over infinite sums;
- A complete idempotent semiring is automatically residuated.

- The semiring $\mathbb{R}_{max}$ is not complete;
- A complete idempotent semiring must have a maximal element;
- It is sufficient to add $+\infty$ (denoted by $\top$) to obtain a complete semiring:

$$\overline{\mathbb{R}}_{max} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}.$$

- Since zero is absorbing, in $\overline{\mathbb{R}}_{max}$ holds

$$\varepsilon \otimes \top = \varepsilon,$$

$$(-\infty) + (+\infty) = (+\infty) + (-\infty) = (-\infty).$$

- $\mathbb{R}_{max}$ can be embedded in the residuated semiring $\overline{\mathbb{R}}_{max}$:

$$a \backslash b = b / a = b - a.$$

Matrix residuation

Let $\in \overline{\mathbb{R}}_{max}^{p\times n}$ and $B \in \overline{\mathbb{R}}_{max}^{p\times m}$.

- By a right residual of B by A we mean the greatest solution of the matrix inequality
$$A \cdot X \leqslant B, \tag{1}$$
where X is an unknown matrix taking values in $\overline{\mathbb{R}}_{max}^{n\times m}$.

Let $A \in \overline{\mathbb{R}}_{max}^{n\times p}$ and $B \in \overline{\mathbb{R}}_{max}^{m\times p}$.

- By a left residual of B by A we mean the greatest solution of the matrix inequality
$$X \cdot A \leqslant B, \tag{2}$$
where X is an unknown matrix taking values in $\overline{\mathbb{R}}_{max}^{m\times n}$.

- Residuals of Boolean valued relations – G. Birkhoff, 1948.
- Approximation to an inverse in the monoid of relations.

- Right residual $A\backslash B$ of B by A

$$(A\backslash B)_{ij} = \bigwedge_{k=1}^{p} A_{ki}\backslash B_{kj}.$$

- Left residual B/A of B by A

$$(B/A)_{ij} = \bigwedge_{k=1}^{p} B_{ik}/A_{jk}.$$

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 2 & 3 & 0 \\ 0 & 4 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 10 \\ 8 \\ 12 \\ 9 \end{bmatrix}.$$

$$A \backslash B = \begin{bmatrix} (1\backslash 10) \wedge (2\backslash 8) \wedge (2\backslash 12) \wedge (0\backslash 9) \\ (0\backslash 10) \wedge (1\backslash 8) \wedge (3\backslash 12) \wedge (4\backslash 9) \\ (2\backslash 10) \wedge (1\backslash 8) \wedge (0\backslash 12) \wedge (1\backslash 9) \end{bmatrix}$$

$$= \begin{bmatrix} 9 \wedge 6 \wedge 10 \wedge 9 \\ 10 \wedge 7 \wedge 9 \wedge 5 \\ 8 \wedge 7 \wedge 12 \wedge 8 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 7 \end{bmatrix}.$$

Weakly linear inequalities

Let $A \in \overline{\mathbb{R}}_{max}^{n\times n}$, $B \in \overline{\mathbb{R}}_{max}^{m\times m}$, $Z \in \overline{\mathbb{R}}_{max}^{n\times m}$ and unknown matrix $X \in \overline{\mathbb{R}}_{max}^{n\times m}$.

$$(1) \quad X^\top \cdot A \leqslant B \cdot X^\top, \quad X \leqslant Z,$$

$$(2) \quad A \cdot X \leqslant X \cdot B, \quad X \leqslant Z.$$

Motivation – in automata theory, study of state reduction, bisimulation and equivalence of automata.

Define functions $\phi^{(t)} : \overline{\mathbb{R}}_{max}^{n\times m} \to \overline{\mathbb{R}}_{max}^{n\times m}$, for $t = 1, 2$:

$$\phi^{(1)}(P) = [B \cdot P^\top / A]^\top,$$

$$\phi^{(2)}(\rho) = A \backslash P \cdot B.$$

A matrix $P \in \overline{\mathbb{R}}_{max}^{n\times m}$ is a solution of weakly linear system (t), for $t = 1, 2$ if and only if it satisfies

$$P \leqslant \phi^t(P) \qquad \text{and} \qquad P \leqslant Z.$$

Let $\{P_k\}_{k\in\mathbb{N}}$ be a sequence of $\overline{\mathbb{R}}_{max}^{n\times m}$ matrices defined inductively by

$$P_1 = Z,$$
$$P_{k+1} = P_k \wedge \phi^{(t)}(P_k).$$

- The sequence $\{P_k\}_{k\in\mathbb{N}}$ is descending.
- If there is the least natural number $m \in \mathbb{N}$ such that $P_m = P_{m+1}$, then the matrix P_m is the greatest matrix which is solution to weakly linear inequality (t).
- If the subalgebra generated by $\{A_{ij}, B_{ij}, Z_{ij}\}$ satisfies DCC, then there is the least natural number $m \in \mathbb{N}$ such that $P_m = P_{m+1}$.

- In some situations we do not need solutions to weakly linear matrix inequalities that are matrices over $\mathbb{R}_{max}$, but those that are Boolean matrices.
- Moreover, in cases where our algorithms for computing the greatest solutions to weakly linears inequalities fail to terminate in a finite number of steps, it is reasonable to search for the greatest Boolean solutions to these systems.
- They can be understood as some kind of "approximations" of the solutions.

For $A \in \overline{\mathbb{R}}_{max}^{p \times n}$ and $B \in \overline{\mathbb{R}}_{max}^{p \times m}$ let a Boolean matrix $A \backslash B \in 2^{n \times m}$ be defined by

$$(A \backslash B)_{ij} = \lceil A_{\downarrow i} \leqslant B_{\downarrow j} \rceil. \tag{3}$$

Then for each $X \in 2^{n \times n}$ the following adjunction property holds

$$A \cdot X \leqslant B \quad \Leftrightarrow \quad X \leqslant A \backslash B, \tag{4}$$

and $A \backslash B$ is the Boolean right residual of B by A.

For $A \in \overline{\mathbb{R}}_{max}^{n \times p}$ and $B \in \overline{\mathbb{R}}_{max}^{m \times p}$ let a Boolean matrix $B/A \in 2^{m \times n}$ be defined by

$$(B/A)_{ji} = \lceil A_{i \rightarrow} \leqslant B_{j \rightarrow} \rceil. \tag{5}$$

Then for each $\xi \in 2^{m \times n}$ the following adjunction property holds

$$X \cdot A \leqslant B \quad \Leftrightarrow \quad X \leqslant B/A, \tag{6}$$

and B/A is the Boolean left residual of B by A.

If we use Boolean right and left residuals:

- ▷ The sequence $\{P_k\}_{k \in \mathbb{N}}$ is finite and descending.
- ▷ There is the least natural number $m \in \mathbb{N}$ such that $P_m = P_{m+1}$.
- ▷ The matrix P_m is the greatest Boolean matrix which is solution to weakly linear inequality (t).

Applications

- Weighted automata theory / Weighted transition systems.
- Damljanović, Ćirić, Ignjatović, TCS 2014.

Weighted finite automaton over an alphabet X and a semiring S is a quadruple

$$\mathcal{A} = (A, \delta^A, \sigma^A, \tau^A),$$

- A is a finite non-empty set of states,
- $\delta^A : A \times X \times A \to S$ is a weighted transition function,
- $\sigma \in S^A$ is an initial weight vector,
- $\tau \in S^A$ is a final weight vector.

For each $x \in X$ we define a weighted transition matrix $\delta_x \in S^{A \times A}$ by

$$\delta_x^A(a, b) = \delta^A(a, x, b) \quad \text{for all } a, b \in A.$$

The behavior of a weighted automaton $\mathcal{A} = (A, \delta^A, \sigma^A, \tau^A)$ is the series $[\![\mathcal{A}]\!]$ defined by

$$([\![\mathcal{A}]\!], u) = \sigma^A \cdot \delta_u^A \cdot \tau^A = \sum_{a_1, a_2 \in A} \sigma^A(a_1) \cdot \delta_u^A(a_1, a_2) \cdot \tau^A(a_2), \quad u \in X^*.$$

Simulations

Let $\mathcal{A} = (A, \delta^A, \sigma^A, \tau^A)$ and $\mathcal{B} = (B, \delta^B, \sigma^B, \tau^B)$ be weighted automata.

A Boolean matrix $\rho \in 2^{A \times B}$ is called a forward simulation between $\mathcal{A}$ and $\mathcal{B}$ if

$$\sigma^A \leqslant \sigma^B \cdot \rho^\top, \tag{fs-1}$$

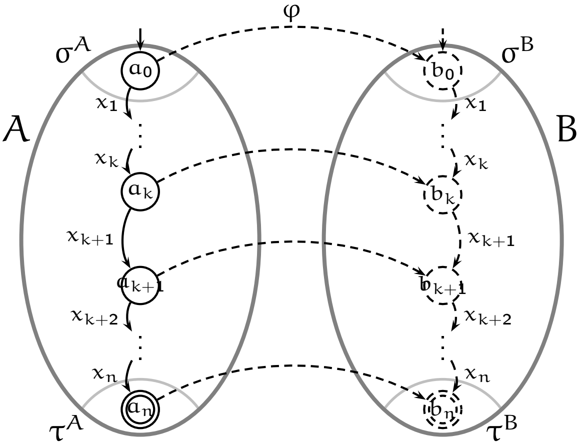
$$\rho^\top \cdot \delta_x^A \leqslant \delta_x^B \cdot \rho^\top, \quad \text{for every } x \in X, \tag{fs-2}$$

$$\rho^\top \cdot \tau^A \leqslant \tau^B. \tag{fs-3}$$

We call ρ a backward simulation between $\mathcal{A}$ and $\mathcal{B}$ if it is a forward simulation between the reverse automata $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$.

Bisimulations

- ρ is forward bisimulation if both ρ and $\rho^\top$ are forward simulations;
- ρ is backward bisimulation, if both ρ and $\rho^\top$ are backward simulations;
- If ρ is a forward simulation and $\rho^\top$ is a backward simulation, then ρ is called a forward-backward bisimulation;
- If ρ is a backward simulation and $\rho^\top$ is a forward simulation, then ρ is called a backward-forward bisimulation.



For an arbitrary successful run $a_0, a_1, \ldots, a_n$ of the automaton $\mathcal{A}$ on a word $u = x_1 x_2 \cdots x_n$ ($x_1, x_2, \ldots, x_n \in X$) we can built a sequence $b_0, b_1, \ldots, b_n$ of states of $\mathcal{B}$ which simulates the original run.

Existence of simulation/bisimulation of a given type between two weighted automata implies behavior inclusion/equivalence between them:

(A) If ρ is a simulation, then $[\![\mathcal{A}]\!] \leqslant [\![\mathcal{B}]\!]$.

(B) If ρ is a bisimulation, then $[\![\mathcal{A}]\!] = [\![\mathcal{B}]\!]$.

Let $\mathcal{A} = (A, \delta^A, \sigma^A, \tau^A)$ and $\mathcal{B} = (B, \delta^B, \sigma^B, \tau^B)$ be weighted automata. A Boolean matrix $\rho \in 2^{A\times B}$ satisfies conditions (fs-2) and (fs-3) if and only if it satisfies

$$\rho \leqslant \bigodot_{x\in X}[(\delta_x^B \cdot \rho^\top)/\delta_x^A]^\top, \qquad \rho \leqslant \tau^A \backslash \tau^B. \tag{7}$$

Now we are ready to prove a theorem which provides a method for testing the existence of a forward simulation between two weighted automata and the construction of the greatest forward simulation, if forward simulations exist.

Let $\mathcal{A} = (A, \delta^A, \sigma^A, \tau^A)$ and $\mathcal{B} = (B, \delta^B, \sigma^B, \tau^B)$ be weighted automata, and let $\{\rho_k\}_{k\in\mathbb{N}} \subseteq 2^{A\times B}$ be a sequence of Boolean matrices defined inductively by

$$\rho_1 = \tau^A \backslash \tau^B, \qquad \rho_{k+1} = \rho_k \odot \left(\bigodot_{x\in X} \left[(\delta_x^B \cdot \rho_k^\top)/\delta_x^A \right]^\top \right), \quad \text{for every } k \in \mathbb{N}. \quad (8)$$

Then the following holds:

(a) The sequence $\{\rho_k\}_{k\in\mathbb{N}}$ is finite and descending, and there is the least natural number $m \in \mathbb{N}$ such that $\rho_m = \rho_{m+1}$;

(b) ρ_m is the greatest Boolean matrix in $2^{A\times B}$ which satisfies $(fs\text{-}2)$ and $(fs\text{-}3)$;

(c) If ρ_m satisfies $(fs\text{-}1)$, then it is the greatest forward simulation between $\mathcal{A}$ and $\mathcal{B}$;

(d) If ρ_m does not satisfy $(fs\text{-}1)$, then there is no forward simulation between $\mathcal{A}$ and $\mathcal{B}$.

Algorithm – Computation of the greatest forward simulation

The input of this algorithm are weighted automata $\mathcal{A} = (A, \delta^A, \sigma^A, \tau^A)$ and $\mathcal{B} = (B, \delta^B, \sigma^B, \tau^B)$. The algorithm decides whether there is a forward simulation between $\mathcal{A}$ and $\mathcal{B}$, and when it exists, the output of the algorithm is the greatest forward simulation.

The procedure is to construct the sequence of Boolean matrices $\{\rho_k\}_{k\in\mathbb{N}}$:

- (A1) In the first step we compute $\tau^A \backslash \tau^B$ and we set $\rho_1 = \tau^A \backslash \tau^B$.
- (A2) After the kth step let ρ_k be the Boolean matrix that has been constructed.
- (A3) In the next step we construct the Boolean matrix ρ_{k+1} by means of the formula (8).
- (A4) Simultaneously, we check whether $\rho_{k+1} = \rho_k$.
- (A5) When we find the smallest number m such that $\rho_{m+1} = \rho_m$, the procedure of constructing the sequence $\{\rho_k\}_{k\in\mathbb{N}}$ terminates, and we check whether ρ_m satisfies $(fs\text{-}1)$. If ρ_m satisfies $(fs\text{-}1)$, then it is the greatest forward simulation between $\mathcal{A}$ and $\mathcal{B}$, and if ρ_m does not satisfy $(fs\text{-}1)$, then there is no any forward simulation between $\mathcal{A}$ and $\mathcal{B}$.

The algorithm terminates in a finite number of steps. The total computation time for the whole algorithm is $O(m|A||B|(|A|+|B|)|X|c_+)$.

Similarly we can give a procedure which decides whether there exists a forward (or backward-forward) bisimulaton between $\mathcal{A}$ and $\mathcal{B}$, and whenever there is at least one such bisimulation, the algorithm computes the greatest one. The only difference is that for forward bisimulations we build the sequence $\{\rho_k\}_{k\in\mathbb{N}}$ by

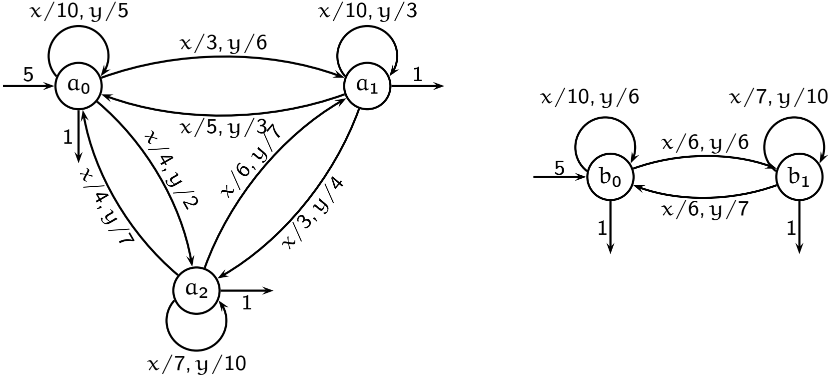
$$\rho_1 = (\tau^A \backslash \tau^B) \odot (\tau^A / \tau^B), \qquad \rho_{k+1} = \rho_k \odot \left(\bigodot_{x\in X} \left[(\delta_x^B \cdot \rho_k^\top) / \delta_x^A \right]^\top \odot \left[(\delta_x^A \cdot \rho_k) / \delta_x^B \right] \right), \tag{9}$$

and at the final stage of the algorithm, we perform the check using conditions $\sigma^A \leqslant \sigma^B \cdot \rho^\top$ and $\sigma^B \leqslant \sigma^A \cdot \rho$ instead of (fs-1), and in the case of backward-forward bisimulations we build $\{\rho_k\}_{k\in\mathbb{N}}$ by

$$\rho_1 = (\sigma^A \backslash \sigma^B) \odot (\tau^A / \tau^B) \qquad \rho_{k+1} = \rho_k \odot \left(\bigodot_{x\in X} \left[(\delta_x^A \cdot \rho_k) / \delta_x^B \right] \odot \left[\delta_x^A \backslash (\rho_k \cdot \delta_x^B) \right] \right), \tag{10}$$

and in the check at the final stage of the algorithm we use conditions $\sigma^B \leqslant \sigma^A \cdot \rho$ and $\tau^A \leqslant \rho \cdot \tau^B$.

Let $\mathcal{A} = (A, \delta^A, \sigma^A, \tau^A)$ and $\mathcal{B} = (B, \delta^B, \sigma^B, \tau^B)$ be weighted automata over an alphabet $X = \{x, y\}$ and the max-plus semiring $(\mathbb{R} \cup \{-\infty\}, \max, +, -\infty, 0)$, with $|A| = 3$ and $|B| = 2$, which are represented by the following graph:



They can also be represented by the following matrices and vectors:

$$\sigma^A = \begin{bmatrix} 5 & 0 & 0 \end{bmatrix}, \quad \delta_x^A = \begin{bmatrix} 10 & 3 & 4 \\ 5 & 10 & 3 \\ 4 & 6 & 7 \end{bmatrix}, \quad \delta_y^A = \begin{bmatrix} 5 & 6 & 2 \\ 3 & 3 & 4 \\ 7 & 7 & 10 \end{bmatrix}, \quad \tau^A = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

$$\sigma^B = \begin{bmatrix} 5 & 0 \end{bmatrix}, \quad \delta_x^B = \begin{bmatrix} 10 & 6 \\ 6 & 7 \end{bmatrix}, \quad \delta_y^B = \begin{bmatrix} 6 & 6 \\ 7 & 10 \end{bmatrix}, \quad \tau^B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Using the above algorithm the following sequence of Boolean matrices has been constructed:

$$\rho_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \rho_3 = \rho_2 = \begin{bmatrix} 0 & -\infty \\ 0 & -\infty \\ -\infty & 0 \end{bmatrix}.$$

The matrix ρ_2 satisfies condition $(fs\text{-}1)$, so it is the greatest forward simulation between $\mathcal{A}$ and $\mathcal{B}$.

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Thank you for your attention!