Action of endomorphism semigroups on definable sets

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joint work with G.Mashevizky and B.Plotkin.

General Reference:

G.Mashevizky, B.Plotkin, E.Plotkin "*Action of endomorphism semigroups on definable sets*", Preprint, 28pp

Setting

Let Θ be a variety of algebras, $F(X)$, $X = \{x_1, \dots, x_n\}$ a free algebra in Θ. Take $A \in \Theta$. A^n the n-th Cartesian power of A can be identified with $Hom(F(X), A)$ where $|X| = n$. Let ϕ be a first order formula. Then

- Every formula ϕ is preserved under isomorphisms of algebras.
- Every formula ϕ without universal quantifiers is preserved under monomorphisms of algebras.
- Every positive formula ϕ is preserved under epimorphisms of algebras.
- Every positive formula ϕ without universal quantifiers is preserved under homomorphisms of algebras.

The converse direction: classical Lyndon's, Los-Tarski's and homomorphism preservation theorems.

- A formula ϕ is preserved under arbitrary monomorphisms of algebras if and only if it is equivalent to a formula without universal quantifiers (Los-Tarski theorem).
- A formula ϕ is preserved under arbitrary epimorphisms of algebras if and only if it is equivalent to a positive formula (Lyndon's positivity theorem).
- A formula ϕ is preserved under arbitrary homomorphisms of algebras if and only it if is equivalent to a positive formula without universal quantifiers (Homomorphism preserving theorem).

Definition

Let K be a set of the first order formulas in a language $\mathbf{L}$. We assign to K the set $\mathbf{G}(K)$ of endomorphisms of A such that any formula $\phi \in K$ is preserved under the action of $\mathbf{G}(K)$.

Definition

Let S be a subsemigroup of $End(A)$. We assign to S the set $\mathbf{G}(S)$ of $\mathbf{L}$-formulas such that any formula $\phi \in \mathbf{G}(S)$ is preserved under the action of S.

Theorem

The correspondence $S \to \mathbf{G}(S)$ and $K \to \mathbf{G}(K)$ between subsemigroups of the endomorphism semigroup $End(A)$ and subsets of first order formulas in $\mathbf{L}$ is the Galois-type correspondence.

Simple properties

1. $\mathbf{G}(K)$ is a subsemigroup of $End(A)$,
2. $\mathbf{G}(K)$ contains $Aut(A)$.

1. $\mathbf{G}(S)$ contains all atomic formulas of the first order language of A.
2. $\mathbf{G}(S)$ is closed under conjunctions $\wedge$ and under disjunctions $\vee$.
3. $\mathbf{G}(S)$ is closed under quantifiers $\exists x$.

Question

Given algebra A, what are the Galois-closed objects?

- Let F_2 be the two-generator free group. Take $\Gamma = F_2 \times F_3^{\mathbb{N}}$, where F_2 and F_3 are free groups. The natural embedding $f : F_2 \to F_3$ is elementary. Then monomorphism $\varphi : \Gamma \to \Gamma$ which sends F_2 to $f(F_2)$ and shifts all other copies of F_3 as above preserves all definable sets. Hence, $\mathbf{G}(\mathbf{L}) \neq Aut(\Gamma)$. The similar examples can be constructed on the base of hyperbolic groups.
- Let $\Gamma = (Z_p)^{\mathbb{N}}$, i.e., Γ is infinite direct product of cyclic groups of prime order. Then a monomorphism $\varphi : \Gamma \to \Gamma$ which shifts the i-th component to the $(i+1)$-th one preserves all definable sets. So, $\mathbf{G}(\mathbf{L}) \neq Aut(\Gamma)$. This example shows that $Aut(A)$ is not necessarily $\mathbf{G}$-closed even for countably categorical algebras.
- Consider the abelian group $A = (\mathbb{Q}, +)$, which is model complete and co-Hopfian. It means that $\mathbf{G}(\mathbf{L}) = Aut(A)$, and $Aut(A)$ is $\mathbf{G}$-closed

$\mathbf{G}(S)$-type of a point $\mu \in A^n$ is the class of all $\mathbf{G}(S)$-formulas which are satisfied by μ and we denote this class by $\mathbf{G}(S)tp(\mu)$.

Definition

Let $S = S(A)$ be a subsemigroup of $End(A)$. Let $\mu \in A^n$ be a point in the affine space.

1. The algebraic S-trace of μ is the set $S\mu = \{\nu = \alpha\mu | \alpha \in S\}$.
2. The algebraic S-orbit of μ is the set $S_{AO}(\mu) = \{\nu | S\mu = S\nu\}$.
3. The logic S-trace of μ is the set $S_{LT}(\mu) = \{\nu | \mathbf{G}(S)tp(\mu) \subset \mathbf{G}(S)tp(\nu)\}$.
4. The logic S-orbit of μ is the set $S_{LO}(\mu) = \{\nu | \mathbf{G}(S)tp(\mu) = \mathbf{G}(S)tp(\nu)\}$.

Definition

An algebra is logically S-homogeneous if $S\mu = S_{LT}(\mu)$.

Definition

An algebra A is S-oligomorphic if there exists only finitely many algebraic S-orbits under the action of S on A^n for any $n \in N$.

Definition

An algebra A is $\mathbf{G}(S)$-atomic if the $\mathbf{G}(S)$-type of any point of A^n is principal.

Theorem

Let S be a submonoid of $End(A)$. The following statements are equivalent.

1. *A is logically S-homogeneous, i.e., $S\mu = S_{LT}(\mu)$.*
2. *$S\mu =< \mu >_S$ for any $\mu \in A^n$.*
3. *$S\mu$ is a $\mathbf{G}(S)$-t-definable set for any $\mu \in A^n$.*
4. *M is the union of $\mathbf{G}(S)$-t-definable sets for any subset M of A^n closed under the action of S.*

The next theorem is a $\mathbf{G}(S)$-version of Ryll-Nardzewski, Engeler and Svenonius result.

Theorem

Let A be an algebra. Let S be a subsemigroup of $End(A)$. The following statements are equivalent.

1. *A possesses the Ryll-Nardzewski $\mathbf{G}(S)$-property.*
2. *A realizes only finitely many n-$\mathbf{G}(S)$-types for each $n \in N$.*
3. *A is $\mathbf{G}(S)$-atomic.*
4. *If A is countable, then A is S-oligomorphic.*

SOME PROBLEMS

Let A be an algebra, **L** a first order language, S a semigroup of endomorphisms of A, i.e., $S \subset End(A)$.

Question

Describe the Galois closed objects for the Galois correspondence **G** (Definitions 1 and 2), i.e., for an algebra A of the given class of algebras (e.g. groups, semigroups, associative algebras) find

- subsemigroups S of $End(A)$ such that $\mathbf{GG}(S) = S$,
- subsemigroups S of $End(A)$ such that $\mathbf{G}_{\overline{L}}\mathbf{G}_{\overline{L}}(S) = S$ holds in some extension $\overline{L}$ of **L**,
- subsets K of **L** such that $\mathbf{GG}(K) = K$.
- Describe the lattice of G-closed subsemigroups of $End(A)$.

- $End(A)$ is the semigroup of all endomorphisms of an algebra A.
- $Aut(A)$ is the group of all automorphisms of an algebra A.
- $SEnd(A)$ is the semigroup of all surjective endomorphisms of an algebra A.
- $IEnd(A)$ is the semigroup of all injective endomorphisms of an algebra A.

Question

Study algebras A such that

- classical semigroups of endomorphisms and sets of formulas are **G**-closed ,
- classical semigroups of endomorphisms and sets of formulas are the only **G**-closed objects,
- in particular, what are the algebras such that any their elementary embedding to itself is an automorphism.

Let Θ be a variety of algebras, $A = F(X)$ the free in Θ algebra over the set $X = \{x_1, \dots, x_t\}$ of free generators. The next problem is related to a well-known question whether the rank of a free algebra is elementary definable.

Question

Given $A = F(X)$, define the subsemigroup $T_k(A)$ of $End(A)$ by $\alpha \in T_k(A)$ if the set $\alpha(X)$ consists of at most k elements. So, $T_k(A) = \{\alpha \in End(A) || \alpha(X)| \leq k\}$. Recall that subsemigroups S_1 and S_2 of $End(A)$ are **G**-equivalent if $\mathbf{GG}(S_1) = \mathbf{GG}(S_2)$.

- Describe $\mathbf{GG}(T_k(A))$. In particular, what are algebras A such that $\mathbf{GG}(T_k(A)) = T_k(A) \cup ElEnd(A)$.
- Describe algebras A such that $T_k(A)$ and $T_s(A)$ are **G**-equivalent for all $k, s \in N$ or for all $k, s \geq m$ for some $m \in N$.