Generalized Attributes in Concept Lattices

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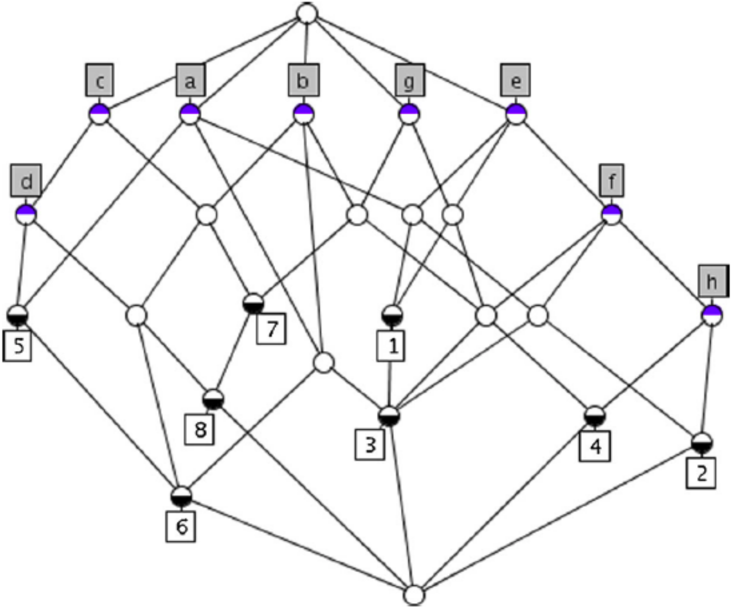
Elementary Information system and Formal Contexts

$\mathbb{K}$	a	b	c	d	e	f	g	h
1	×				×		×	
2	×				×	×		×
3	×	×			×	×	×	
4		×			×	×	×	×
5	×		×	×				
6	×	×	×	×				
7		×	×				×	
8		×	×	×			×	

- A **context** is a triple $\mathbb{K} := (G, M, I)$ with sets G (of objects), M (of attributes) and $I \subseteq G \times M$ a binary relation.
- A **concept** is a pair (A, B) with B the set of all properties common to objects in A and A the set of all objects having all the properties in B.

Lattice of concepts



Lattice of concepts (FCA)

- **Context**: $\mathbb{K} := (G, M, \mathrm{I})$ with $\mathrm{I} \subseteq G \times M$.
- $g \,\mathrm{I}\, m :\iff (g, m) \in \mathrm{I}$. $\qquad$ g has attribute m.

$$A' := \{m \in M \mid \forall g \in A \; g \,\mathrm{I}\, m\} \quad \text{and} \quad B' := \{g \in G \mid \forall m \in B \; g \,\mathrm{I}\, m\}.$$

- A **formal concept** of $\mathbb{K}$ is a pair (A, B) with $A' = B$ and $B' = A$.
- A is the **extent** and B the **intent** of the concept (A, B).

- $c : X \mapsto X''$ is a closure operator on $\mathcal{P}(G)$ and on $\mathcal{P}(M)$.
- $\mathrm{Ext}(\mathbb{K}) := c(\mathcal{P}(G)) \cong^d c(\mathcal{P}(M)) =: \mathrm{Int}(\mathbb{K})$.

- $\mathfrak{B}(\mathbb{K}) :=$ set of all formal concepts of $\mathbb{K}$.
- Concept hierarchy: $(A, B) \leq (C, D)$ iff $A \subseteq C$ (iff $D \subseteq B$).
- $(\mathfrak{B}(\mathbb{K}); \leq)$ is a complete lattice, called **concept lattice** of $\mathbb{K}$.

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Generalized patterns

- In data mining, generalized patterns are pieces of knowledge extracted from data when an ontology is used. For example the attributes of $\mathbb{K}$ can be grouped together to form set S of new attributes.
- In basket market analysis, items or products can be grouped into product lines or product categories. Customers may be grouped according to some specific features (e.g., income, education).
- By grouping the attributes of $\mathbb{K}$, we actually replace (G, M, I) with a new context (G, S, J) with S covering M and J to be precised.
- There are mainly three ways to express the relation J:
  1. gJs :iff g has at least one attribute from the group s
  2. gJs :iff g has all attributes from the group s
  3. gJs :iff g satisfies at least a certain proportion of the attributes in s

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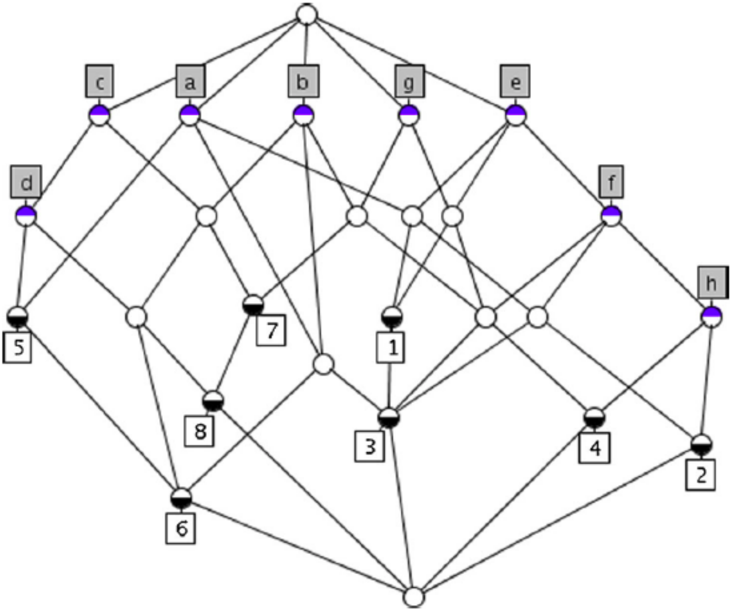

Generalizing attributes

Initial context									∃-generalization				∀-generalization				α-generalization		
	a	b	c	d	e	f	g	h	A	B	C	D	S	T	U	V	E	F	H
1	×				×		×		×		×		×						
2	×				×	×		×	×		×	×				×		×	
3	×	×			×	×	×		×	×	×	×	×				×	×	
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5	×		×	×						×	×				×		×		
6	×	×	×	×						×	×			×	×		×		
7		×	×					×		×	×			×			×		
8		×	×	×			×		×	×	×			×			×		

The generalized attributes are

(∃) $A := \{e, g\}$, $B := \{b, c\}$, $C := \{a, d\}$, $D := \{f, h\}$.

(∀) $S := \{e, g\}$, $T := \{b, c\}$, $U := \{a, d\}$, $V := \{f, h\}$.

(α) $E := \{a, b, c\}$, $F := \{d, e, f\}$, $H := \{g, h\}$ with threshold $\alpha = 60\%$

Expected Gain: We reduce the size of the context and expect also the size of the concept lattice to reduce. BUT this is not always the case.

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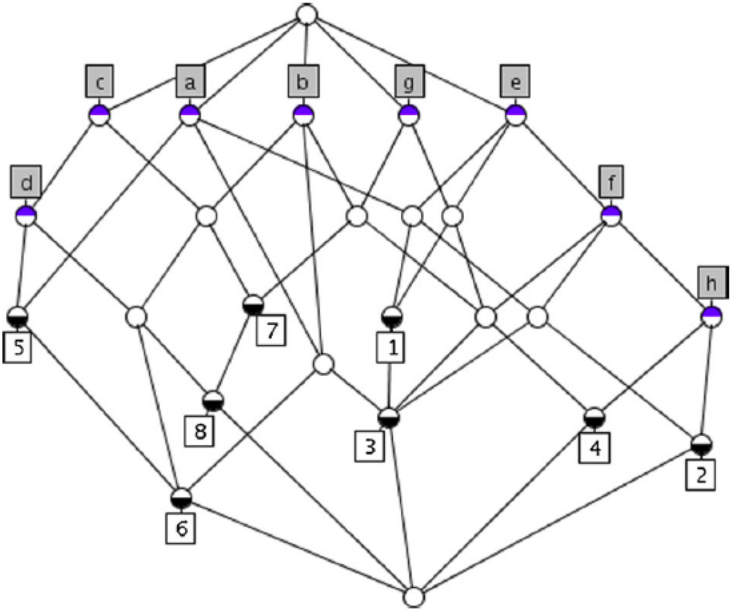
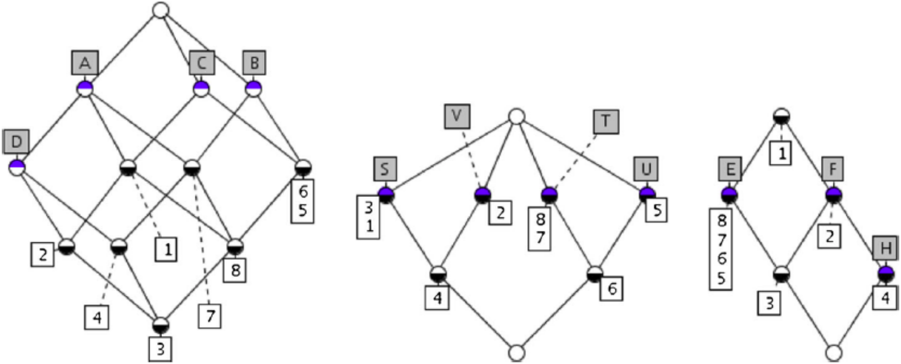
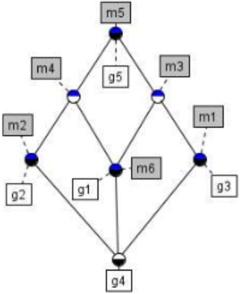


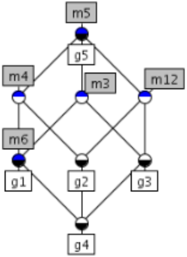
Lattice of concepts with generalized attributes

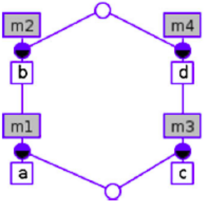


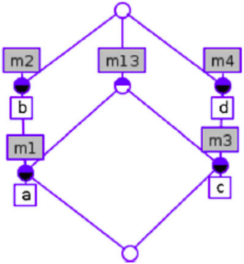
For this example the lattice size decreases in all three cases.

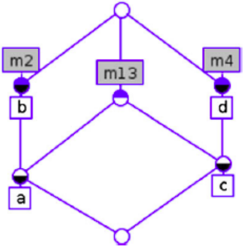
∃-Generalizing two attributes can increase the lattice size!











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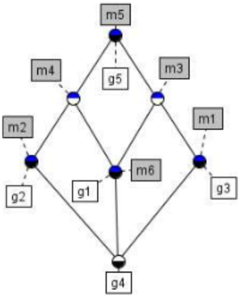

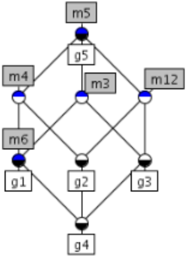

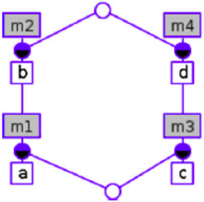

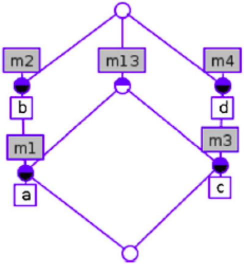

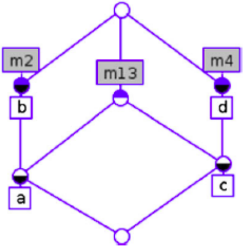

Observation and questions

- The ∀-generalizations on attributes do not increase the size of the concept lattice.
- If the concept lattice is distributive, then any ∃-generalization reduces the size of the initial lattice.
- The lattice B_4 is the smallest lattice on which there is an ∃-generalization that increases the size of the initial concept lattice.

Questions

1. Can the size increase by more than one after a ∃-generalisation?
2. Can the size remains unchanged after a ∃-generalisation?
3. Can we characterize contexts for which the size does not decrease after a ∃-generalization? e.g in terms of forbidden configurations?
4. Is there a similarity measure (on attributes) compatible with the changing of size after a generalization?

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Adding one attribute to a context

- Generalizing two attributes $a, b \in M$ in (G, M, I) is done by adding an attribute $ab \notin M$ to M and removing a, b from M.
- Let $\mathbb{K}_m = (G, M \cup \{s\}, \mathrm{I}_s)$ be an extension of $\mathbb{K} = (G, M, \mathrm{I})$, $s \notin M$.
- Let $(A, B) \in \mathfrak{B}(\mathbb{K})$;
  - If $A \subseteq m'$ then $(A, B \cup \{m\}) \in \mathfrak{B}(\mathbb{K}_m)$.
  - Else, (A, B) and $(A \cap m', A' \cup \{m\})$ are two different concepts of $\mathbb{K}_m$.
- The map

$$\Phi_m : (A, B) \mapsto \begin{cases} (A, B \cup \{m\}) : & \text{if } A \subseteq m' \\ (A, B) & \text{else} \end{cases}$$

  is injective and order preserving from $\mathfrak{B}(\mathbb{K})$ to $\mathfrak{B}(\mathbb{K}_\mathfrak{m})$.
- $\Delta_m = |\mathfrak{B}(\mathbb{K}_\mathfrak{m})| - |\mathfrak{B}(\mathbb{K})| \leq |\mathfrak{B}(\mathbb{K})|$.
  The equality can be reached.

The size can increase exponentially!

- $\mathbb{K} = (S, S, \neq)$ has $2^{|S|}$ concepts, that form a Boolean algebra.
- Let $S_n = \{1, \ldots, n\}$, $n \in \mathbb{N}_*$ and $g_1, m_1, m_2 \notin S_n$.
- Set $\mathbb{K}_n^k := (S_n \dot\cup \{g_1\}, S_n \dot\cup \{m_1, m_2\}, \mathrm{I})$ with $k \in S_n$ and
  - $\mathrm{I} \cap (S_n \times S_n) = \neq$
  - $g_1' = S_n$, $m_1' = \{1, \ldots, k\}$ and $m_2' = S_n \setminus m_1'$.

- The context resulting from a $\exists$-generalization of m_1 and m_2 is isomorphic to $(S_{n+1}, S_{n+1}, \neq)$ and therefore has 2^{n+1} concepts.
- The context $\mathbb{K}_n^k$ has $2^n + 2^k + 2^{n-k} - 1$ concepts.
- Putting m_1 and m_2 together increases the size by $2^n - 2^k - 2^{n-k} + 1$.
- The maximal increase arise with $k = \frac{n}{2}$ if n is even, or with $k \in \left\{\lfloor\frac{n}{2}\rfloor, \lceil\frac{n}{2}\rceil\right\}$ if n is odd.

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Is this the worst case?

$$g(k) = 2^n - 2^k - 2^{n-k} + 1$$
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$$g''(k) = -\ln^2(2)2^k - \ln^2(2)2^{n-k} < 0.$$

- For any context (G, M, I), the number of concept is $\leq 2^{\min(|M|,|G|)}$.
- Let $(G, M \cup \{a, b\}, I)$ be a context and $(G, M \cup \{ab\}, I)$ the context obtained by $\exists$-generalizing a and b.
- $\begin{cases} |\mathfrak{B}(G, M, I)| \leq |\mathfrak{B}(G, M \cup \{a, b\}, I)| \\ |\mathfrak{B}(G, M, I)| \leq |\mathfrak{B}(G, M \cup \{ab\}, I)| \end{cases}$
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- Set $\begin{cases} n_a := |\mathfrak{B}(G, M \cup \{a\}, I)| - |\mathfrak{B}(G, M, I)| \\ n_{a+b} := |\mathfrak{B}(G, M \cup \{a, b\}, I)| - |\mathfrak{B}(G, M, I)| \\ n_{ab} := |\mathfrak{B}(G, M \cup \{ab\}, I)| - |\mathfrak{B}(G, M, I)| \end{cases}$

Is this the worst case?

$$g(k) = 2^n - 2^k - 2^{n-k} + 1$$
$$0 = g'(k) = -\ln(2)2^k + \ln(2)2^{n-k} \iff n = 2k.$$
$$g''(k) = -\ln^2(2)2^k - \ln^2(2)2^{n-k} < 0.$$

- For any context (G, M, I), the number of concept is $\leq 2^{\min(|M|,|G|)}$.
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- Let $(G, M \cup \{a, b\}, I)$ be a context. $\exists$-Generalizing a and b increases the concept lattice size iff $n_{ab} > n_{a+b}$.
- The map

$$\begin{array}{cccc} \phi_a : & \mathfrak{B}(G, M, I) & \longrightarrow & \mathfrak{B}(G, M \cup \{a\}, I) \\ & (A, B) & \longmapsto & \begin{cases} (A, B \cup \{a\}) & \text{if } A \subseteq a' \\ (A, B) & \text{else} \end{cases} \end{array}$$

  is an injective map.
- If $a' = G$ then Φ_a is a bijection.
- If a is reducible (i.e. $\exists Y \subseteq M$ such that $a' = Y'$) then Φ_a is a bijection.

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- The maximum for n_a is $2^{|a'|}$ if all $A \cap a'$ are distinct extents of $\mathbb{K}$.
- So n_{ab} is maximal if all $A \cap (a' \cup b')$ are distinct extents of (G, M, I).
- The order ideal generated by $\{\mu a, \mu b\}$ is then isomorphic to $\mathcal{P}(a' \cup b') \setminus \{a' \cup b'\}$.
- For a reduced context (G, M, I) the choice for n_{ab} to reach the max is with $|M| - 1 = |ab'| = |a' \cup b'|$.
- The increase n_{a+b} after adding both a and b is minimal when $a' \cap b' = \emptyset$ holds. That is $n_{a+b} = n_a + n_b - 1$
- Thus $n_{ab} - n_{a+b} \leq 2^{|a'|+|b'|} - 2^{|a'|} - 2^{|b'|} + 1$

The maximal increase after $\exists$-generalizing is reached when n_{ab} is maximal and n_{a+b} minimal and is $n_{ab} - n_{a+b} \leq 2^{|a'|+|b'|} - 2^{|a'|} - 2^{|b'|} + 1$