Generalized Attributes in Concept Lattices

Léonard Kwuida

Bern University of Applied Sciences, Switzerland

June 15, 2017

with R. Kuitché and E. R. Temgoua

UYI and ENS, Yaoundé - Cameroon
Elementary Information system and Formal Contexts

A context is a triple $\mathbb{K} := (G, M, I)$ with sets $G$ (of objects), $M$ (of attributes) and $I \subseteq G \times M$ a binary relation.

A concept is a pair $(A, B)$ with $B$ the set of all properties common to objects in $A$ and $A$ the set of all objects having all the properties in $B$. 
Lattice of concepts
Lattice of concepts (FCA)

- **Context:** $\mathbb{K} := (G, M, I)$ with $I \subseteq G \times M$.
- $g \mid m : \iff (g, m) \in I$. 
  $g$ has attribute $m$.

\[ A' := \{ m \in M \mid \forall g \in A \ g \mid m \} \quad \text{and} \quad B' := \{ g \in G \mid \forall m \in B \ g \mid m \}. \]

- A formal concept of $\mathbb{K}$ is a pair $(A, B)$ with $A' = B$ and $B' = A$.
- $A$ is the extent and $B$ the intent of the concept $(A, B)$.

- $c : X \mapsto X''$ is a closure operator on $\mathcal{P}(G)$ and on $\mathcal{P}(M)$.
- $\text{Ext}(\mathbb{K}) := c(\mathcal{P}(G)) \cong c(\mathcal{P}(M)) =: \text{Int}(\mathbb{K})$.

- $\mathcal{B}(\mathbb{K}) := \text{set of all formal concepts of } \mathbb{K}$.
- Concept hierarchy: $(A, B) \leq (C, D)$ iff $A \subseteq C$ (iff $D \subseteq B$).
- $(\mathcal{B}(\mathbb{K}); \leq)$ is a complete lattice, called concept lattice of $\mathbb{K}$.
Lattice of concepts (FCA)

- **Context**: $\mathbb{K} := (G, M, I)$ with $I \subseteq G \times M$.
- $g I m : \iff (g, m) \in I$. $g$ has attribute $m$.

$$A' := \{m \in M \mid \forall g \in A \ g I m\} \quad \text{and} \quad B' := \{g \in G \mid \forall m \in B \ g I m\}.$$

- A **formal concept** of $\mathbb{K}$ is a pair $(A, B)$ with $A' = B$ and $B' = A$.
- $A$ is the **extent** and $B$ the **intent** of the concept $(A, B)$.

- $c : X \mapsto X''$ is a closure operator on $\mathcal{P}(G)$ and on $\mathcal{P}(M)$.
- $\text{Ext}(\mathbb{K}) := c(\mathcal{P}(G)) \cong d c(\mathcal{P}(M)) =: \text{Int}(\mathbb{K})$.

- $\mathfrak{B}(\mathbb{K}) := \text{set of all formal concepts of } \mathbb{K}$.
- Concept hierarchy: $(A, B) \leq (C, D)$ iff $A \subseteq C$ (iff $D \subseteq B$).
- $(\mathfrak{B}(\mathbb{K}); \leq)$ is a complete lattice, called **concept lattice** of $\mathbb{K}$.
Generalized patterns

- In data mining, generalized patterns are pieces of knowledge extracted from data when an ontology is used. For example the attributes of \( \mathbb{K} \) can be grouped together to form set \( S \) of new attributes.

- In basket market analysis, items or products can be grouped into product lines or product categories. Customers may be grouped according to some specific features (e.g., income, education).

- By grouping the attributes of \( \mathbb{K} \), we actually replace \((G, M, I)\) with a new context \((G, S, J)\) with \( S \) covering \( M \) and \( J \) to be precised.

There are mainly three ways to express the relation \( J \):

1. \( gJ_s : \text{iff } g \text{ has at least one attribute from the group } s \)
2. \( gJ_s : \text{iff } g \text{ has all attributes from the group } s \)
3. \( gJ_s : \text{iff } g \text{ satisfies at least a certain proportion of the attributes in } s \)
Generalized patterns

- In data mining, generalized patterns are pieces of knowledge extracted from data when an ontology is used. For example the attributes of $K$ can be grouped together to form set $S$ of new attributes.

- In basket market analysis, items or products can be grouped into product lines or product categories. Customers may be grouped according to some specific features (e.g., income, education).

- By grouping the attributes of $K$, we actually replace $(G, M, I)$ with a new context $(G, S, J)$ with $S$ covering $M$ and $J$ to be precised.

- There are mainly three ways to express the relation $J$:
  1. $gJ_s$: iff $g$ has at least one attribute from the group $s$
  2. $gJs$: iff $g$ has all attributes from the group $s$
  3. $gJs$: iff $g$ satisfies at least a certain proportion of the attributes in $s$
Generalized patterns

- In data mining, generalized patterns are pieces of knowledge extracted from data when an ontology is used. For example the attributes of $\mathbb{K}$ can be grouped together to form set $S$ of new attributes.

- In basket market analysis, items or products can be grouped into product lines or product categories. Customers may be grouped according to some specific features (e.g., income, education).

- By grouping the attributes of $\mathbb{K}$, we actually replace $(G, M, I)$ with a new context $(G, S, J)$ with $S$ covering $M$ and $J$ to be precised.

- There are mainly three ways to express the relation $J$:
  1. $gJs :\text{iff } g$ has at least one attribute from the group $s$
  2. $gJs :\text{iff } g$ has all attributes from the group $s$
  3. $gJs :\text{iff } g$ satisfies at least a certain proportion of the attributes in $s$
Lattice of concepts
The generalized attributes are

\( (\exists) \ A := \{e, g\}, \ B := \{b, c\}, \ C := \{a, d\}, \ D := \{f, h\}. \)

\( (\forall) \ S := \{e, g\}, \ T := \{b, c\}, \ U := \{a, d\}, \ V := \{f, h\}. \)

\( (\alpha) \ E := \{a, b, c\}, \ F := \{d, e, f\}, \ H := \{g, h\} \) with threshold \( \alpha = 60\% \)
Generalizing attributes

The generalized attributes are

$(\exists) \ A := \{e, g\}, \ B := \{b, c\}, \ C := \{a, d\}, \ D := \{f, h\}.$

$(\forall) \ S := \{e, g\}, \ T := \{b, c\}, \ U := \{a, d\}, \ V := \{f, h\}.$

$(\alpha) \ E := \{a, b, c\}, \ F := \{d, e, f\}, \ H := \{g, h\}$ with threshold $\alpha = 60\%$

**Expected Gain:** We reduce the size of the context and expect also the size of the concept lattice to reduce. BUT this is not always the case.
Lattice of concepts
Lattice of concepts with generalized attributes

For this example the lattice size decreases in all three cases.
∃-Generalizing two attributes can increase the lattice size!
∃ - Generalizing two attributes can increase the lattice size!
Observation and questions

- The $\forall$-generalizations on attributes do not increase the size of the concept lattice.
- If the concept lattice is distributive, then any $\exists$-generalization reduces the size of the initial lattice.
- The lattice $B_4$ is the smallest lattice on which there is an $\exists$-generalization that increases the size of the initial concept lattice.

Questions

1. Can the size increase by more than one after a $\exists$-generalisation?
2. Can the size remains unchanged after a $\exists$-generalisation?
3. Can we characterize contexts for which the size does not decrease after a $\exists$-generalization? e.g in terms of forbidden configurations?
4. Is there a similarity measure (on attributes) compatible with the changing of size after a generalization?
Observation and questions

- The $\forall$-generalizations on attributes do not increase the size of the concept lattice.
- If the concept lattice is distributive, then any $\exists$-generalization reduces the size of the initial lattice.
- The lattice $B_4$ is the smallest lattice on which there is an $\exists$-generalization that increases the size of the initial concept lattice.

Questions

1. Can the size increase by more than one after a $\exists$-generalisation?
2. Can the size remains unchanged after a $\exists$-generalisation?
3. Can we characterize contexts for which the size does not decrease after a $\exists$-generalization? e.g in terms of forbidden configurations?
4. Is there a similarity measure (on attributes) compatible with the changing of size after a generalization?
Observation and questions

- The $\forall$-generalizations on attributes do not increase the size of the concept lattice.
- If the concept lattice is distributive, then any $\exists$-generalization reduces the size of the initial lattice.
- The lattice $B_4$ is the smallest lattice on which there is an $\exists$-generalization that increases the size of the initial concept lattice.

Questions

1. Can the size increase by more than one after a $\exists$-generalisation?
2. Can the size remain unchanged after a $\exists$-generalisation?
3. Can we characterize contexts for which the size does not decrease after a $\exists$-generalization? e.g in terms of forbidden configurations?
4. Is there a similarity measure (on attributes) compatible with the changing of size after a generalization?
Observation and questions

- The $\forall$-generalizations on attributes do not increase the size of the concept lattice.
- If the concept lattice is distributive, then any $\exists$-generalization reduces the size of the initial lattice.
- The lattice $B_4$ is the smallest lattice on which there is an $\exists$-generalization that increases the size of the initial concept lattice.

Questions

1. Can the size increase by more than one after a $\exists$-generalisation?
2. Can the size remains unchanged after a $\exists$-generalisation?
3. Can we characterize contexts for which the size does not decrease after a $\exists$-generalization? e.g. in terms of forbidden configurations?
4. Is there a similarity measure (on attributes) compatible with the changing of size after a generalization?
Adding one attribute to a context

- Generalizing two attributes \( a, b \in M \) in \((G, M, I)\) is done by adding an attribute \( ab \notin M \) to \( M \) and removing \( a, b \) from \( M \).
- Let \( \mathbb{K}_m = (G, M \cup \{s\}, I_s) \) be an extension of \( \mathbb{K} = (G, M, I), s \notin M \).
- Let \((A, B) \in \mathcal{B}(\mathbb{K})\);
  - If \( A \subseteq m' \) then \((A, B \cup \{m\}) \in \mathcal{B}(\mathbb{K}_m)\).
  - Else, \((A, B)\) and \((A \cap m', A' \cup \{m\})\) are two different concepts of \( \mathbb{K}_m \).
- The map

\[
\Phi_m : (A, B) \mapsto \begin{cases} 
(A, B \cup \{m\}) : & \text{if } A \subseteq m' \\
(A, B) : & \text{else}
\end{cases}
\]

is injective and order preserving from \( \mathcal{B}(\mathbb{K}) \) to \( \mathcal{B}(\mathbb{K}_m) \).
- \( \Delta_m = |\mathcal{B}(\mathbb{K}_m)| - |\mathcal{B}(\mathbb{K})| \leq |\mathcal{B}(\mathbb{K})| \).
  The equality can be reached.
The size can increase exponentially!

- $K = (S, S, \neq)$ has $2^{|S|}$ concepts, that form a Boolean algebra.
- Let $S_n = \{1, \ldots, n\}$, $n \in \mathbb{N}_*$ and $g_1, m_1, m_2 \not\in S_n$.
- Set $K^k_n := (S_n \cup \{g_1\}, S_n \cup \{m_1, m_2\}, I)$ with $k \in S_n$ and
  - $I \cap (S_n \times S_n) \neq$
  - $g'_1 = S_n$, $m'_1 = \{1, \ldots, k\}$ and $m'_2 = S_n \setminus m'_1$.

- The context resulting from a $\exists$-generalization of $m_1$ and $m_2$ is isomorphic to $(S_{n+1}, S_{n+1}, \neq)$ and therefore has $2^{n+1}$ concepts.
- The context $K^k_n$ has $2^n + 2^k + 2^{n-k} - 1$ concepts.
- Putting $m_1$ and $m_2$ together increases the size by $2^n - 2^k - 2^{n-k} + 1$.
- The maximal increase arise with $k = \frac{n}{2}$ if $n$ is even, or with $k \in \{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil\}$ if $n$ is odd.
The size can increase exponentially!

- \( K = (S, S, \neq) \) has \( 2^{|S|} \) concepts, that form a Boolean algebra.
- Let \( S_n = \{1, \ldots, n\} \), \( n \in \mathbb{N}_* \) and \( g_1, m_1, m_2 \notin S_n \).
- Set \( K_n^k := (S_n \cup \{g_1\}, S_n \cup \{m_1, m_2\}, I) \) with \( k \in S_n \) and
  - \( I \cap (S_n \times S_n) \neq \)
  - \( g_1' = S_n, \quad m_1' = \{1, \ldots, k\} \) and \( m_2' = S_n \setminus m_1' \).

- The context resulting from a \( \exists \)-generalization of \( m_1 \) and \( m_2 \) is isomorphic to \((S_{n+1}, S_{n+1}, \neq)\) and therefore has \( 2^{n+1} \) concepts.
- The context \( K_n^k \) has \( 2^n + 2^k + 2^{n-k} - 1 \) concepts.
- Putting \( m_1 \) and \( m_2 \) together increases the size by \( 2^n - 2^k - 2^{n-k} + 1 \).
- The maximal increase arise with \( k = \frac{n}{2} \) if \( n \) is even, or with \( k \in \left\{ \lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil \right\} \) if \( n \) is odd.
The size can increase exponentially!

- $\mathbb{K} = (S, S, \neq)$ has $2^{|S|}$ concepts, that form a Boolean algebra.
- Let $S_n = \{1, \ldots, n\}$, $n \in \mathbb{N}_*$ and $g_1, m_1, m_2 \notin S_n$.
- Set $\mathbb{K}_n^k := (S_n \cup \{g_1\}, S_n \cup \{m_1, m_2\}, I)$ with $k \in S_n$ and
  - $I \cap (S_n \times S_n) \neq \emptyset$
  - $g_1' = S_n$, $m_1' = \{1, \ldots, k\}$ and $m_2' = S_n \setminus m_1'$.

- The context resulting from a $\exists$-generalization of $m_1$ and $m_2$ is isomorphic to $(S_{n+1}, S_{n+1}, \neq)$ and therefore has $2^{n+1}$ concepts.
- The context $\mathbb{K}_n^k$ has $2^n + 2^k + 2^{n-k} - 1$ concepts.
- Putting $m_1$ and $m_2$ together increases the size by $2^n - 2^k - 2^{n-k} + 1$.
- The maximal increase arise with $k = \frac{n}{2}$ if $n$ is even, or with $k \in \left\{\left\lfloor \frac{n}{2} \right\rfloor, \left\lceil \frac{n}{2} \right\rceil\right\}$ if $n$ is odd.
The size can increase exponentially!

- $K = (S, S, \neq)$ has $2^{|S|}$ concepts, that form a Boolean algebra.
- Let $S_n = \{1, \ldots, n\}, \ n \in \mathbb{N}_*$ and $g_1, m_1, m_2 \not\in S_n$.
- Set $K^n_k := (S_n \cup \{g_1\}, S_n \cup \{m_1, m_2\}, I)$ with $k \in S_n$ and
  - $I \cap (S_n \times S_n) \neq \emptyset$
  - $g'_1 = S_n, \ m'_1 = \{1, \ldots, k\}$ and $m'_2 = S_n \setminus m'_1$.

- The context resulting from a $\exists$-generalization of $m_1$ and $m_2$ is isomorphic to $(S_{n+1}, S_{n+1}, \neq)$ and therefore has $2^{n+1}$ concepts.
- The context $K^n_k$ has $2^n + 2^k + 2^{n-k} - 1$ concepts.
- Putting $m_1$ and $m_2$ together increases the size by $2^n - 2^k - 2^{n-k} + 1$.
- The maximal increase arise with $k = \frac{n}{2}$ if $n$ is even, or with $k \in \left\{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil \right\}$ if $n$ is odd.
The size can increase exponentially!

- $\mathbb{K} = (S, S, \neq)$ has $2^{|S|}$ concepts, that form a Boolean algebra.
- Let $S_n = \{1, \ldots, n\}$, $n \in \mathbb{N}_*$ and $g_1, m_1, m_2 \notin S_n$.
- Set $\mathbb{K}_n^k := (S_n \cup \{g_1\}, S_n \cup \{m_1, m_2\}, I)$ with $k \in S_n$ and
  - $I \cap (S_n \times S_n) \neq \emptyset$
  - $g'_1 = S_n$, $m'_1 = \{1, \ldots, k\}$ and $m'_2 = S_n \setminus m'_1$.

- The context resulting from a $\exists$-generalization of $m_1$ and $m_2$ is isomorphic to $(S_{n+1}, S_{n+1}, \neq)$ and therefore has $2^{n+1}$ concepts.
- The context $\mathbb{K}_n^k$ has $2^n + 2^k + 2^{n-k} - 1$ concepts.
- Putting $m_1$ and $m_2$ together increases the size by $2^n - 2^k - 2^{n-k} + 1$.
- The maximal increase arise with $k = \frac{n}{2}$ if $n$ is even, or with $k \in \left\{ \left\lfloor \frac{n}{2} \right\rfloor, \left\lceil \frac{n}{2} \right\rceil \right\}$ if $n$ is odd.
Is this the worst case?

\[ g(k) = 2^n - 2^k - 2^{n-k} + 1 \]

\[ 0 = g'(k) = -\ln(2)2^k + \ln(2)2^{n-k} \iff n = 2k. \]

\[ g''(k) = -\ln^2(2)2^k - \ln^2(2)2^{n-k} < 0. \]

- For any context \((G, M, I)\), the number of concept is \(\leq 2^{\min(|M|, |G|)}\).
- Let \((G, M \cup \{a, b\}, I)\) be a context and \((G, M \cup \{ab\}, I)\) the context obtained by \(\exists\)-generalizing \(a\) and \(b\).

\[
\begin{cases}
|\mathcal{B}(G, M, I)| & \leq |\mathcal{B}(G, M \cup \{a, b\}, I)| \\
|\mathcal{B}(G, M, I)| & \leq |\mathcal{B}(G, M \cup \{ab\}, I)|
\end{cases}
\]

\[
\begin{cases}
    n_a := |\mathcal{B}(G, M \cup \{a\}, I)| - |\mathcal{B}(G, M, I)| \\
    n_{a+b} := |\mathcal{B}(G, M \cup \{a, b\}, I)| - |\mathcal{B}(G, M, I)| \\
    n_{ab} := |\mathcal{B}(G, M \cup \{ab\}, I)| - |\mathcal{B}(G, M, I)|
\end{cases}
\]
Is this the worst case?

\[ g(k) = 2^n - 2^k - 2^{n-k} + 1 \]

\[ 0 = g'(k) = -\ln(2)2^k + \ln(2)2^{n-k} \iff n = 2k. \]

\[ g''(k) = -\ln^2(2)2^k - \ln^2(2)2^{n-k} < 0. \]

- For any context \((G, M, I)\), the number of concept is \(\leq 2^{\min(|M|, |G|)}\).
- Let \((G, M \cup \{a, b\}, I)\) be a context and \((G, M \cup \{ab\}, I)\) the context obtained by \(\exists\)-generalizing \(a\) and \(b\).

\[
\begin{cases}
|B(G, M, I)| & \leq |B(G, M \cup \{a, b\}, I)| \\
|B(G, M, I)| & \leq |B(G, M \cup \{ab\}, I)|
\end{cases}
\]

Set \[
\begin{align*}
    n_a & := |B(G, M \cup \{a\}, I)| - |B(G, M, I)| \\
    n_{a+b} & := |B(G, M \cup \{a, b\}, I)| - |B(G, M, I)| \\
    n_{ab} & := |B(G, M \cup \{ab\}, I)| - |B(G, M, I)|
\end{align*}
\]

Kwuida (BUAS)
Generalized attributes
Novi Sad 2017 14 / 17
Is this the worst case?

\[ g(k) = 2^n - 2^k - 2^{n-k} + 1 \]

\[ 0 = g'(k) = -\ln(2)2^k + \ln(2)2^{n-k} \iff n = 2k. \]

\[ g''(k) = -\ln^2(2)2^k - \ln^2(2)2^{n-k} < 0. \]

- For any context \((G, M, I)\), the number of concept is \(\leq 2^{\min(|M|,|G|)}\).
- Let \((G, M \cup \{a, b\}, I)\) be a context and \((G, M \cup \{ab\}, I)\) the context obtained by \(\exists\)-generalizing \(a\) and \(b\).

\[
\begin{cases} 
|\mathcal{B}(G, M, I)| \leq |\mathcal{B}(G, M \cup \{a, b\}, I)| \\
|\mathcal{B}(G, M, I)| \leq |\mathcal{B}(G, M \cup \{ab\}, I)| 
\end{cases}
\]

- Set \(n_a := |\mathcal{B}(G, M \cup \{a\}, I)| - |\mathcal{B}(G, M, I)|\)
- Set \(n_{a+b} := |\mathcal{B}(G, M \cup \{a, b\}, I)| - |\mathcal{B}(G, M, I)|\)
- Set \(n_{ab} := |\mathcal{B}(G, M \cup \{ab\}, I)| - |\mathcal{B}(G, M, I)|\)
Is this the worst case?

\[ g(k) = 2^n - 2^k - 2^{n-k} + 1 \]
\[ 0 = g'(k) = -\ln(2)2^k + \ln(2)2^{n-k} \iff n = 2k. \]
\[ g''(k) = -\ln^2(2)2^k - \ln^2(2)2^{n-k} < 0. \]

- For any context \((G, M, I)\), the number of concept is \(\leq 2^{\min(|M|, |G|)}\).
- Let \((G, M \cup \{a, b\}, I)\) be a context and \((G, M \cup \{ab\}, I)\) the context obtained by \(\exists\)-generalizing \(a\) and \(b\).

\[
\begin{align*}
|\mathcal{B}(G, M, I)| &\leq |\mathcal{B}(G, M \cup \{a, b\}, I)|, \\
|\mathcal{B}(G, M, I)| &\leq |\mathcal{B}(G, M \cup \{ab\}, I)|.
\end{align*}
\]

- Set

\[
\begin{align*}
 n_a &:= |\mathcal{B}(G, M \cup \{a\}, I)| - |\mathcal{B}(G, M, I)|, \\
n_{a+b} &:= |\mathcal{B}(G, M \cup \{a, b\}, I)| - |\mathcal{B}(G, M, I)|, \\
n_{ab} &:= |\mathcal{B}(G, M \cup \{ab\}, I)| - |\mathcal{B}(G, M, I)|.
\end{align*}
\]
Let \((G, M \cup \{a, b\}, I)\) be a context. \(\exists\)-Generalizing \(a\) and \(b\) increases the concept lattice size iff \(n_{ab} > n_{a+b}\).

The map

\[
\phi_a : \mathcal{B}(G, M, I) \rightarrow \mathcal{B}(G, M \cup \{a\}, I)
\]

\[
(A, B) \mapsto \begin{cases} 
(A, B \cup \{a\}) & \text{if } A \subseteq a' \\
(A, B) & \text{else}
\end{cases}
\]

is an injective map.

If \(a' = G\) then \(\Phi_a\) is a bijection.

If \(a\) is reducible (i.e. \(\exists Y \subseteq M\) such that \(a' = Y'\)) then \(\Phi_a\) is a bijection.
Let \((G, M \cup \{a, b\}, I)\) be a context. \(\exists\)-Generalizing \(a\) and \(b\) increases the concept lattice size iff \(n_{ab} > n_{a+b}\).

The map

\[
\phi_a : \mathcal{B}(G, M, I) \rightarrow \mathcal{B}(G, M \cup \{a\}, I)
\]

\[
(A, B) \rightarrow \begin{cases} 
(A, B \cup \{a\}) & \text{if } A \subseteq a' \\
(A, B) & \text{else}
\end{cases}
\]

is an injective map.

- If \(a' = G\) then \(\Phi_a\) is a bijection.
- If \(a\) is reducible (i.e. \(\exists Y \subseteq M\) such that \(a' = Y'\)) then \(\Phi_a\) is a bijection.
Let \((G, M \cup \{a, b\}, I)\) be a context. \(\exists\)-Generalizing \(a\) and \(b\) increases the concept lattice size iff \(n_{ab} > n_{a+b}\).

The map

\[
\phi_a : \mathcal{B}(G, M, I) \longrightarrow \mathcal{B}(G, M \cup \{a\}, I)
\]

\[
(A, B) \longmapsto \begin{cases} (A, B \cup \{a\}) & \text{if } A \subseteq a' \\ (A, B) & \text{else} \end{cases}
\]

is an injective map.

- If \(a' = G\) then \(\Phi_a\) is a bijection.
- If \(a\) is reducible (i.e. \(\exists Y \subseteq M\) such that \(a' = Y'\)) then \(\Phi_a\) is a bijection.
Let \((G, M \cup \{a, b\}, I)\) be a context. \(\exists\)-Generalizing \(a\) and \(b\) increases the concept lattice size iff \(n_{ab} > n_{a+b}\).

The map

\[
\phi_a : \mathcal{B}(G, M, I) \longrightarrow \mathcal{B}(G, M \cup \{a\}, I)
\]

\[
(A, B) \mapsto \begin{cases} 
(A, B \cup \{a\}) & \text{if } A \subseteq a' \\
(A, B) & \text{else}
\end{cases}
\]

is an injective map.

If \(a' = G\) then \(\Phi_a\) is a bijection.

If \(a\) is reducible (i.e. \(\exists Y \subseteq M\) such that \(a' = Y'\)) then \(\Phi_a\) is a bijection.
The maximum for $n_a$ is $2|a'|$ if all $A \cap a'$ are distinct extents of $K$.

So $n_{ab}$ is maximal if all $A \cap (a' \cup b')$ are distinct extents of $(G, M, I)$.

The order ideal generated by $\{\mu a, \mu b\}$ is then isomorphic to $\mathcal{P}(a' \cup b') \setminus \{a' \cup b'\}$.

For a reduced context $(G, M, I)$ the choice for $n_{ab}$ to reach the max is with $|M| - 1 = |ab'| = |a' \cup b'|$.

The increase $n_{a+b}$ after adding both $a$ and $b$ is minimal when $a' \cap b' = \emptyset$ holds. That is $n_{a+b} = n_a + n_b - 1$.

Thus $n_{ab} - n_{a+b} \leq 2|a'| + |b'| - 2|a'| - 2|b'| + 1$.
The maximal increase after \( \exists \)-generalizing is reached when \( n_{ab} \) is maximal and \( n_{a+b} \) minimal and is 
\[
 n_{ab} - n_{a+b} \leq 2|a'| + |b'| - 2|a'| - 2|b'| + 1
\]