On partial matroids, geometric posets and semimodular posets

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- **atomistic** if it contains the bottom and each element is a join of atoms;
- **semimodular** if for all $x, y \in L$;

$$x \wedge y \prec x \quad \text{implies} \quad y \prec x \vee y.$$

- **geometric** if it is atomistic, semimodular and such that all chains in L are finite.

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A **matroid** is defined as a set A together with a closure operator $\overline{\cdot} : \mathcal{P}(A) \to \mathcal{P}(A)$ on A such that for all $X \subseteq A$ and for all $x, y \in A$ we have

M_1 : $x \notin \overline{X}$ and $x \in \overline{X \cup \{y\}}$ imply $y \in \overline{X \cup \{x\}}$;

M_2 : there exists a finite Y such that $Y \subseteq X$ and $\overline{Y} = \overline{X}$.

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For every finite matroid there exists a simple matroid such that their lattices of flats (closed subsets) are isomorphic.

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Theorem (Šešelja, Tepavčević; 2000)

The range of a partial closure operator on a set S is a centralized system. Conversely, for every centralized system $\mathcal{F}$ on S, there is a partial closure operator on S such that its range is $\mathcal{F}$.

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A partial closure operator C on S is said to be **sharp** (SPCO), if it satisfies condition:

P_{C_5}: Let $B \subseteq S$. If $\bigcap\{X \in \mathcal{F}_C \mid B \subseteq X\} \in \mathcal{F}_C$, then $C(B)$ is defined and

$$C(B) = \bigcap\{X \in \mathcal{F}_C \mid B \subseteq X\} \qquad \text{(sharpness)}.$$

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Theorem (Šešelja, S., Tepavčević; 2017)

The range of a partial closure operator on a set S is a centralized system. Conversely, for every centralized system $\mathcal{F}$ on S, there is a **unique** *sharp partial closure operator on S such that its range is $\mathcal{F}$.*

Geometric posets

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A lattice of finite length is geometric if and only if it is atomistic and, for every two atoms a and b and any element x, if $a < x \vee b$ and $a \nleq x$, then $b < x \vee a$.

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- If $(P, \leqslant)$ does not have the least element, then we extend the notion of **atoms** of P to all minimal elements of P.
- A_P = the set of all atoms
- A poset is **atomistic** if every element different from the least is supremum of a set of atoms.

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Theorem (Šešelja, S., Tepavčević; 2017)

A poset $(P, \leqslant)$ is geometric if and only if P is atomistic and

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$$\text{then } x \vee y \text{ exists and } y \prec x \vee y.$$

Partial matroids

Definition

We define a **partial matroid** *(p-***matroid***) as a pair* (E, C), *where* E *is a nonempty set and* C *a sharp partial closure operator (SPCO) on* E, *satisfying the following conditions: for every* $X \subseteq E$,

(M) *if* $C(X)$ *and* $C(X \cup \{x\})$ *are defined, then the relations* $y \notin C(X)$ *and* $y \in C(X \cup \{x\})$ *imply that* $C(X \cup \{y\})$ *is defined and* $x \in C(X \cup \{y\})$;

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Theorem (Šešelja, S., Tepavčević; 2017)

The range of a p-matroid with respect to set inclusion is a geometric poset.

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Proof (sketch). $(P, \leqslant)$ - a geometric poset, A its set of atoms.

A partial mapping $C : \mathcal{P}(A) \to \mathcal{P}(A)$:

$$C(X) := \begin{cases} \{a \in A \mid a \leqslant \bigvee X\} & , \quad \text{if } \bigvee X \text{ exists;} \\ \text{not defined} & , \quad \text{otherwise.} \end{cases}$$

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- C is a sharp partial closure operator
- (A, C) is a p-matroid
- $(P, \leqslant)$ and $(\mathcal{F}_C, \subseteq)$ are isomorphic:

$$f : P \to \mathcal{F}_C \text{ defined by } f(x) = \{a \in A \mid a \leqslant x\} \qquad \blacksquare$$

Examples.

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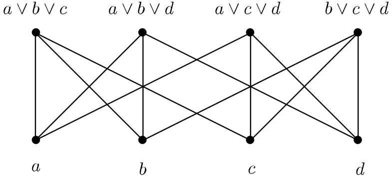
- Geometric poset $(G_{4,3}, \leqslant)$ and the corresponding p-matroid (E, C), where $E = \{a, b, c, d\}$ and

$$C : \begin{pmatrix} \{a\} & \{b\} & \{c\} & \{d\} & \{a,b,c\} & \{a,b,d\} & \{a,c,d\} & \{b,c,d\} \\ \{a\} & \{b\} & \{c\} & \{d\} & \{a,b,c\} & \{a,b,d\} & \{a,c,d\} & \{b,c,d\} \end{pmatrix}$$



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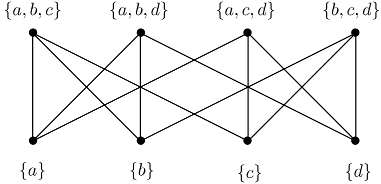

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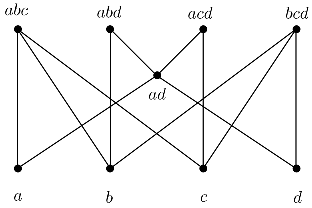
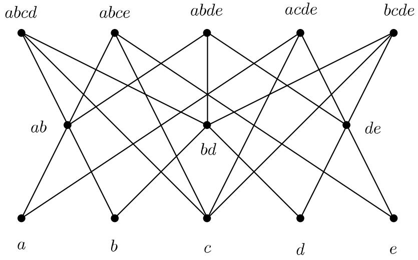
- Two more examples





Semimodular posets

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- *A poset* $(P, \leqslant)$ *which has the least element is* **semimodular** *if for every* $x, y \in P$ *the following holds:*

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- *A poset* $(P, \leqslant)$ *which does not have the least element is* **semimodular** *if the poset* $(P_0, \leqslant)$ *is semimodular, where* $(P_0, \leqslant)$ *is the poset obtained by adding the least element to the poset* $(P, \leqslant)$*.*

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Proposition

Let $(L, \leqslant)$ be a lattice. Then L is semimodular as a lattice if and only if it is semimodular as a poset.

Semimodular posets

(O. Ore, 1943). A poset P is (upper) **semimodular** if it satisfies: if $a \neq b$ both cover c, then there is $d \in P$ which covers both a and b.

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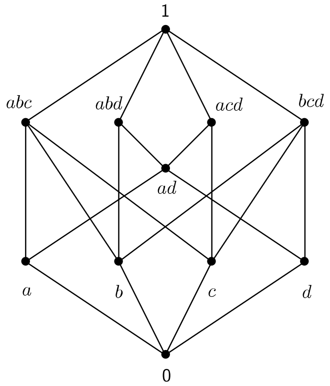

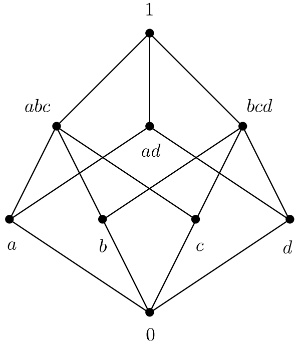

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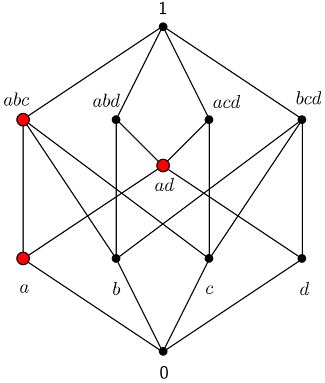

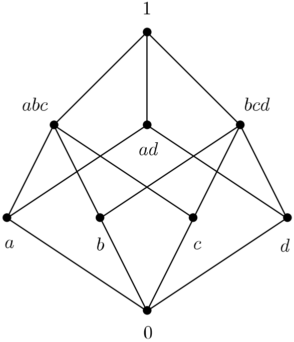

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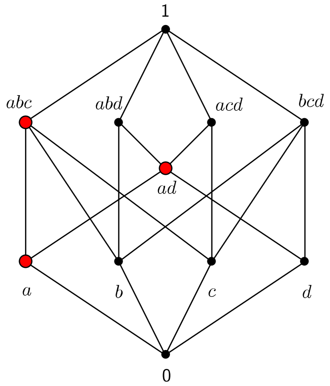

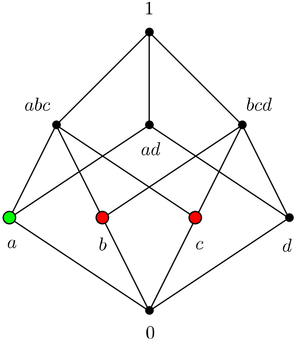

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Corollary (Šešelja, S., Tepavčević; 2017)

A poset is geometric if and only if it is atomistic and semimodular.

Conclusion

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matroid

set + closure operator + $M1$,$M2$,$M3$

$\updownarrow$

geometric lattice

atomistic + semimodular lattice

Conclusion

matroid

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p-matroid

set + sharp partial closure operator

+ M, P

geometric lattice

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***p*-matroid**

set + sharp partial closure operator + M, P

geometric lattice

atomistic + semimodular lattice

geometric poset

atomistic + semimodular poset

Conclusion

matroid

set + closure operator + $M1, M2, M3$

***p*-matroid**

set + sharp partial closure operator + M, P

geometric lattice

atomistic + semimodular lattice

geometric poset

atomistic + semimodular poset

References

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Thank you for your attention!