On the structure of finite commutative totally ordered monoids

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Outline

- Introduction
  - Totally ordered monoids (Tomonoids)
- Tomonoid partitions
- Finite negative tomonoids
  - One-element Rees co-extensions
- General finite tomonoids

Totally ordered monoid

Monoid ... $(S; \oplus, 0)$

- S ... set
- $\oplus$ ... associative binary operation on S
- 0 ... neutral (unit) element

Tomonoid (totally ordered monoid) ... $(S; \leqslant, \oplus, 0)$

- $\leqslant$ ... total (linear) order on S that is *compatible* with $\oplus$:

$$a \leq b \quad \text{implies} \quad a \oplus c \leq b \oplus c \quad \text{and} \quad c \oplus a \leq c \oplus b$$

for every $a, b, c \in S$

Totally ordered monoid

A tomonoid $(S; \leqslant, \oplus, 0)$ is called

finite	...	if S is finite
negative	...	if $0 = \top$
positive	...	if $0 = \bot$
commutative	...	if $a \oplus b = b \oplus a$ for every $a, b \in S$
Archimedean	...	if $\top$, 0, and $\bot$ are the only idempotents

- $a \in S$ is *idempotent* if $a \oplus a = a$

Examples

$(\text{-}8, \text{-}7, \text{-}6, \text{-}5, \text{-}4, \text{-}3, \text{-}2, \text{-}1, 0)$

0	-8	-7	-6	-5	-4	-3	-2	-1	0
-1	-8	-8	-8	-5	-4	-3	-2	-1	-1
-2	-8	-8	-8	-5	-4	-4	-2	-2	-2
-3	-8	-8	-8	-8	-8	-8	-5	-3	-3
-4	-8	-8	-8	-8	-8	-8	-5	-4	-4
-5	-8	-8	-8	-8	-8	-8	-5	-5	-5
-6	-8	-8	-8	-8	-8	-8	-8	-8	-6
-7	-8	-8	-8	-8	-8	-8	-8	-8	-7
-8	-8	-8	-8	-8	-8	-8	-8	-8	-8
	-8	-7	-6	-5	-4	-3	-2	-1	0

- negative
- -1 and -2 are idempotents
- non-commutative
- non-Archimedean

$(\text{-}2, \text{-}1, 0, 1, 2, 3, 4, 5, 6)$

6	6	6	6	6	6	6	6	6	6
5	4	4	5	5	5	5	5	5	6
4	4	4	4	5	5	5	5	5	6
3	1	2	3	4	4	4	5	5	6
2	1	1	2	4	4	4	5	5	6
1	1	1	1	4	4	4	5	5	6
0	-2	-1	0	1	2	3	4	5	6
-1	-2	-2	-1	1	1	2	4	4	6
-2	-2	-2	-2	1	1	1	4	4	6
	-2	-1	0	1	2	3	4	5	6

- non-negative
- non-positive
- commutative
- Archimedean

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Level set representation of a tomonoid

- $(S; \leqslant, \oplus, 0)$ ... tomonoid
- *level equivalence* ... binary relation $\sim$ on $S \times S = S^2$:

$$(a,b) \sim (c,d) \quad \text{if} \quad a \oplus b = c \oplus d$$

- $(S^2; \trianglelefteq, \sim, (0{,}0))$ ... *tomonoid partition* of S

$$(a,b) \trianglelefteq (c,d) \quad \text{if} \quad a \leq c \quad \text{and} \quad b \leq d$$

Example

	-3	-2	-1	0
0	-3	-2	-1	0
-1	-3	-3	-1	-1
-2	-3	-3	-2	-2
-3	-3	-3	-3	-3

equivalence classes of $\sim$

- $\{(0,0)\}$
- $\{(-1,0), (-1,-1), (0,-1)\}$
- $\{(-2,0), (-2,-1), (0,-2)\}$
- $\{(-3,0), (-3,-1), (-3,-2), (-3,-3), (-2,-3), (-1,-3), (0,-3), (-2,-2), (-1,-2)\}$

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-3	-3	-3	-3	-3

equivalence classes of $\sim$

- $\{(0,0)\}$
- $\{(\text{-}1,0),(\text{-}1,\text{-}1),(0,\text{-}1)\}$
- $\{(\text{-}2,0),(\text{-}2,\text{-}1),(0,\text{-}2)\}$
- $\{(\text{-}3,0),(\text{-}3,\text{-}1),(\text{-}3,\text{-}2),(\text{-}3,\text{-}3),(\text{-}2,\text{-}3),$
  $(\text{-}1,\text{-}3),(0,\text{-}3),(\text{-}2,\text{-}2),(\text{-}1,\text{-}2)\}$

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Level set representation of a tomonoid

Level equivalence $\sim$ has the following properties:

(P1) For every $a, b, c, d, e \in S$, $(a, b) \sim (d, 0)$ and $(b, c) \sim (0, e)$ imply $(d, c) \sim (a, e)$.

- associativity

(P2) For every $a, b \in S$ there is exactly one $c \in S$ such that $(a, b) \sim (0, c) \sim (c, 0)$.

- neutral element

(P3) For every $a, b, c, d, a', b', c', d' \in S$, $(a, b) \sim (a', b') \trianglelefteq (c, d) \sim (c', d') \trianglelefteq (a, b)$ implies $(a, b) \sim (c, d)$.

- compatibility with the order

$(S^2; \trianglelefteq, \sim, (0,0))$ satisfies (P1)–(P3) $\Leftrightarrow$ it is a tomonoid partition

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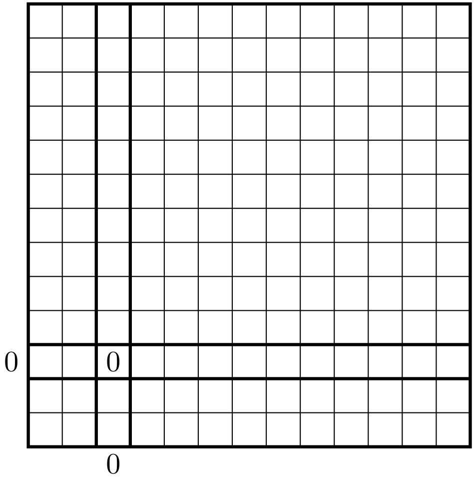
Condition (P1)

(P1) $(a,b) \sim (d,0)$ and $(b,c) \sim (0,e)$ imply $(d,c) \sim (a,e)$



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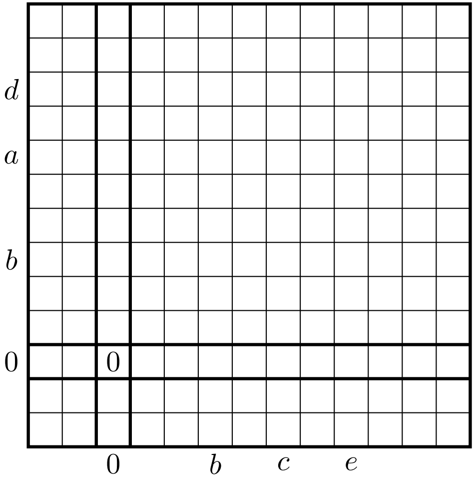

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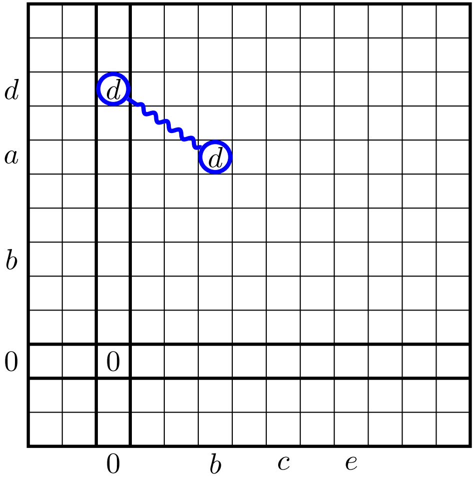

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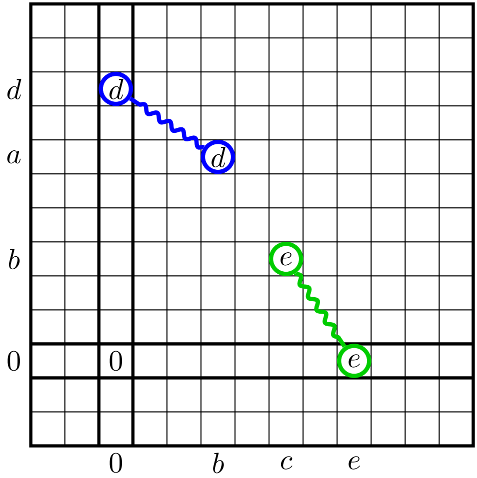

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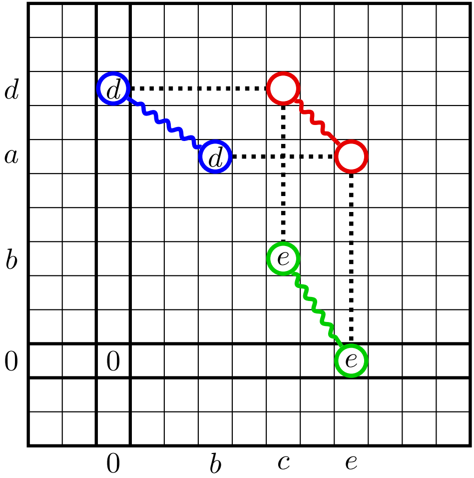

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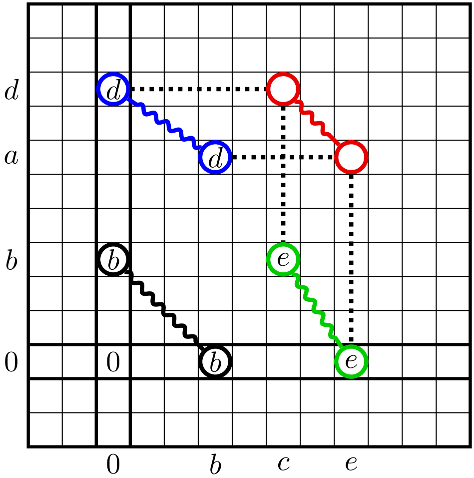

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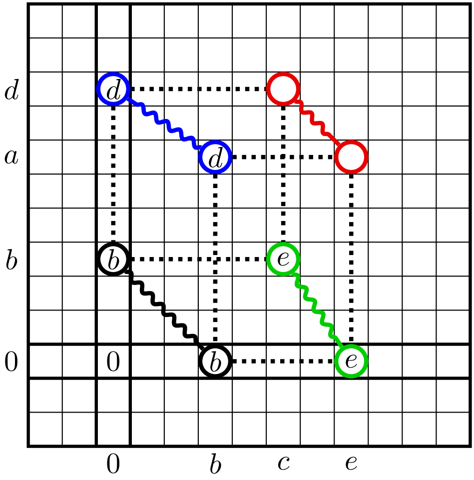

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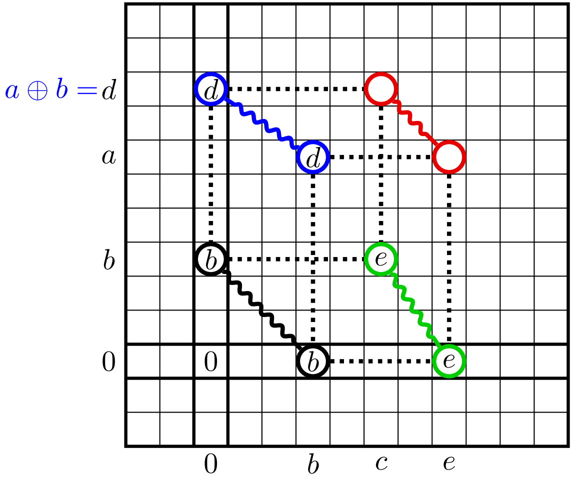

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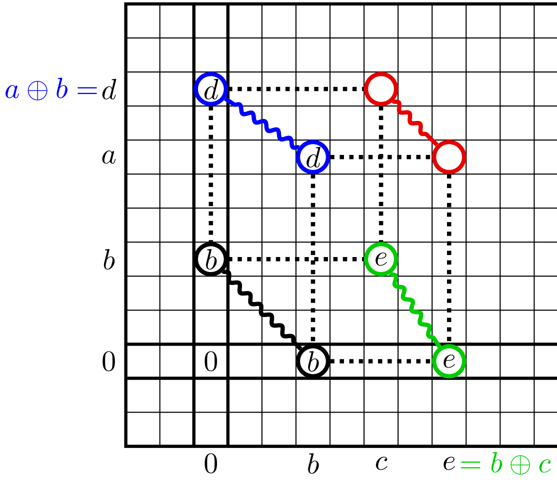

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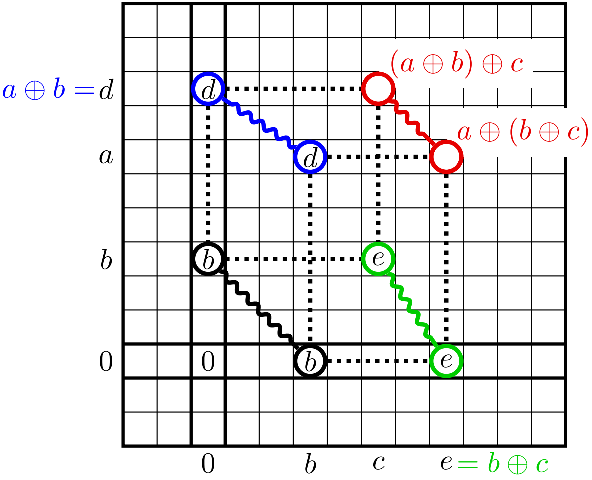

Reidemeister condition

- Condition (P1) corresponds with the *Reidemeister closure condition* known from *web geometry* where it characterizes those loops that are associative (and thus they are groups).

W. Blaschke, G. Bol, *Geometrie der Gewebe, topologische Fragen der Differentialgeometrie*, Springer, 1939.

J. Aczél, *Quasigroups, nets and nomograms*, Advances in Mathematics, 1965.

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Rees quotients

- $(S; \leqslant, \oplus, 0)$ ... finite negative tomonoid
- for given $q \in S$ define binary relation $\approx_q$ on S by

$$x \approx_q y \quad \text{if} \quad x = y \quad \text{or} \quad x, y \leq q$$

- $\approx_q$ ... *Rees congruence* (with respect to q)
- $S/_{\approx_q}$ ... *Rees quotient* of S (with respect to q)

- if $q = \alpha$ (atom of S)

  $\Rightarrow$ $S/_{\approx_\alpha}$ is called *one-element Rees quotient* of S

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$$S: \qquad 0 < u < v < w < x < y < z < 1$$

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Chain of one-element Rees quotients

-7	-6	-5	-4	-3	-2	-1	0
-7	-7	-5	-4	-3	-2	-1	-1
-7	-7	-5	-4	-4	-2	-2	-2
-7	-7	-7	-7	-7	-5	-3	-3
-7	-7	-7	-7	-7	-5	-4	-4
-7	-7	-7	-7	-7	-5	-5	-5
-7	-7	-7	-7	-7	-7	-7	-6
-7	-7	-7	-7	-7	-7	-7	-7

-6	-5	-4	-3	-2	-1	0
-6	-5	-4	-3	-2	-1	-1
-6	-5	-4	-4	-2	-2	-2
-6	-6	-6	-6	-5	-3	-3
-6	-6	-6	-6	-5	-4	-4
-6	-6	-6	-6	-5	-5	-5
-6	-6	-6	-6	-6	-6	-6

-5	-4	-3	-2	-1	0
-5	-4	-3	-2	-1	-1
-5	-4	-4	-2	-2	-2
-5	-5	-5	-5	-3	-3
-5	-5	-5	-5	-4	-4
-5	-5	-5	-5	-5	-5

-4	-3	-2	-1	0
-4	-3	-2	-1	-1
-4	-4	-2	-2	-2
-4	-4	-4	-3	-3
-4	-4	-4	-4	-4

-3	-2	-1	0
-3	-2	-1	-1
-3	-2	-2	-2
-3	-3	-3	-3

-2	-1	0
-2	-1	-1
-2	-2	-2

-2	0
-2	-2

0

Chain of one-element Rees quotients

-7	-6	-5	-4	-3	-2	-1	0
-7	-7	-5	-4	-3	-2	-1	-1
-7	-7	-5	-4	-4	-2	-2	-2
-7	-7	-7	-7	-7	-5	-3	-3
-7	-7	-7	-7	-7	-5	-4	-4
-7	-7	-7	-7	-7	-5	-5	-5
-7	-7	-7	-7	-7	-7	-7	-6
-7	-7	-7	-7	-7	-7	-7	-7

→

-6	-5	-4	-3	-2	-1	0
-6	-5	-4	-3	-2	-1	-1
-6	-5	-4	-4	-2	-2	-2
-6	-6	-6	-6	-5	-3	-3
-6	-6	-6	-6	-5	-4	-4
-6	-6	-6	-6	-5	-5	-5
-6	-6	-6	-6	-6	-6	-6

→

-5	-4	-3	-2	-1	0
-5	-4	-3	-2	-1	-1
-5	-4	-4	-2	-2	-2
-5	-5	-5	-5	-3	-3
-5	-5	-5	-5	-4	-4
-5	-5	-5	-5	-5	-5

→

-4	-3	-2	-1	0
-4	-3	-2	-1	-1
-4	-4	-2	-2	-2
-4	-4	-4	-3	-3
-4	-4	-4	-4	-4

→

-3	-2	-1	0
-3	-2	-1	-1
-3	-2	-2	-2
-3	-3	-3	-3

→

-2	-1	0
-2	-1	-1
-2	-2	-2

→

-2	0
-2	-2

→

0

Chain of one-element Rees quotients

-7	-6	-5	-4	-3	-2	-1	0
-7	-7	-5	-4	-3	-2	-1	-1
-7	-7	-5	-4	-4	-2	-2	-2
-7	-7	-7	-7	-7	-5	-3	-3
-7	-7	-7	-7	-7	-5	-4	-4
-7	-7	-7	-7	-7	-5	-5	-5
-7	-7	-7	-7	-7	-7	-7	-6
-7	-7	-7	-7	-7	-7	-7	-7

→

-6	-5	-4	-3	-2	-1	0
-6	-5	-4	-3	-2	-1	-1
-6	-5	-4	-4	-2	-2	-2
-6	-6	-6	-6	-5	-3	-3
-6	-6	-6	-6	-5	-4	-4
-6	-6	-6	-6	-5	-5	-5
-6	-6	-6	-6	-6	-6	-6

→

-5	-4	-3	-2	-1	0
-5	-4	-3	-2	-1	-1
-5	-4	-4	-2	-2	-2
-5	-5	-5	-5	-3	-3
-5	-5	-5	-5	-4	-4
-5	-5	-5	-5	-5	-5

→

-4	-3	-2	-1	0
-4	-3	-2	-1	-1
-4	-4	-2	-2	-2
-4	-4	-4	-3	-3
-4	-4	-4	-4	-4

→

-3	-2	-1	0
-3	-2	-1	-1
-3	-2	-2	-2
-3	-3	-3	-3

→

-2	-1	0
-2	-1	-1
-2	-2	-2

→

-2	0
-2	-2

→

0

Chain of one-element Rees quotients

-7	-6	-5	-4	-3	-2	-1	0
-7	-7	-5	-4	-3	-2	-1	-1
-7	-7	-5	-4	-4	-2	-2	-2
-7	-7	-7	-7	-7	-5	-3	-3
-7	-7	-7	-7	-7	-5	-4	-4
-7	-7	-7	-7	-7	-5	-5	-5
-7	-7	-7	-7	-7	-7	-7	-6
-7	-7	-7	-7	-7	-7	-7	-7

→

-6	-5	-4	-3	-2	-1	0
-6	-5	-4	-3	-2	-1	-1
-6	-5	-4	-4	-2	-2	-2
-6	-6	-6	-6	-5	-3	-3
-6	-6	-6	-6	-5	-4	-4
-6	-6	-6	-6	-5	-5	-5
-6	-6	-6	-6	-6	-6	-6

→

-5	-4	-3	-2	-1	0
-5	-4	-3	-2	-1	-1
-5	-4	-4	-2	-2	-2
-5	-5	-5	-5	-3	-3
-5	-5	-5	-5	-4	-4
-5	-5	-5	-5	-5	-5

↓

-4	-3	-2	-1	0
-4	-3	-2	-1	-1
-4	-4	-2	-2	-2
-4	-4	-4	-3	-3
-4	-4	-4	-4	-4

←

-3	-2	-1	0
-3	-2	-1	-1
-3	-2	-2	-2
-3	-3	-3	-3

←

-2	-1	0
-2	-1	-1
-2	-2	-2

←

-2	0
-2	-2

←

0

Chain of one-element Rees quotients

-7	-6	-5	-4	-3	-2	-1	0
-7	-7	-5	-4	-3	-2	-1	-1
-7	-7	-5	-4	-4	-2	-2	-2
-7	-7	-7	-7	-7	-5	-3	-3
-7	-7	-7	-7	-7	-5	-4	-4
-7	-7	-7	-7	-7	-5	-5	-5
-7	-7	-7	-7	-7	-7	-7	-6
-7	-7	-7	-7	-7	-7	-7	-7

-6	-5	-4	-3	-2	-1	0
-6	-5	-4	-3	-2	-1	-1
-6	-5	-4	-4	-2	-2	-2
-6	-6	-6	-6	-5	-3	-3
-6	-6	-6	-6	-5	-4	-4
-6	-6	-6	-6	-5	-5	-5
-6	-6	-6	-6	-6	-6	-6

-5	-4	-3	-2	-1	0
-5	-4	-3	-2	-1	-1
-5	-4	-4	-2	-2	-2
-5	-5	-5	-5	-3	-3
-5	-5	-5	-5	-4	-4
-5	-5	-5	-5	-5	-5

-4	-3	-2	-1	0
-4	-3	-2	-1	-1
-4	-4	-2	-2	-2
-4	-4	-4	-3	-3
-4	-4	-4	-4	-4

-3	-2	-1	0
-3	-2	-1	-1
-3	-2	-2	-2
-3	-3	-3	-3

-2	-1	0
-2	-1	-1
-2	-2	-2

-2	0
-2	-2

0

Chain of one-element Rees quotients

-7	-6	-5	-4	-3	-2	-1	0
-7	-7	-5	-4	-3	-2	-1	-1
-7	-7	-5	-4	-4	-2	-2	-2
-7	-7	-7	-7	-7	-5	-3	-3
-7	-7	-7	-7	-7	-5	-4	-4
-7	-7	-7	-7	-7	-5	-5	-5
-7	-7	-7	-7	-7	-7	-7	-6
-7	-7	-7	-7	-7	-7	-7	-7

→

-6	-5	-4	-3	-2	-1	0
-6	-5	-4	-3	-2	-1	-1
-6	-5	-4	-4	-2	-2	-2
-6	-6	-6	-6	-5	-3	-3
-6	-6	-6	-6	-5	-4	-4
-6	-6	-6	-6	-5	-5	-5
-6	-6	-6	-6	-6	-6	-6

→

-5	-4	-3	-2	-1	0
-5	-4	-3	-2	-1	-1
-5	-4	-4	-2	-2	-2
-5	-5	-5	-5	-3	-3
-5	-5	-5	-5	-4	-4
-5	-5	-5	-5	-5	-5

↓

-4	-3	-2	-1	0
-4	-3	-2	-1	-1
-4	-4	-2	-2	-2
-4	-4	-4	-3	-3
-4	-4	-4	-4	-4

←

-3	-2	-1	0
-3	-2	-1	-1
-3	-2	-2	-2
-3	-3	-3	-3

←

-2	-1	0
-2	-1	-1
-2	-2	-2

←

-2	0
-2	-2

←

0

Chain of one-element Rees quotients

-7	-6	-5	-4	-3	-2	-1	0
-7	-7	-5	-4	-3	-2	-1	-1
-7	-7	-5	-4	-4	-2	-2	-2
-7	-7	-7	-7	-7	-5	-3	-3
-7	-7	-7	-7	-7	-5	-4	-4
-7	-7	-7	-7	-7	-5	-5	-5
-7	-7	-7	-7	-7	-7	-7	-6
-7	-7	-7	-7	-7	-7	-7	-7

→

-6	-5	-4	-3	-2	-1	0
-6	-5	-4	-3	-2	-1	-1
-6	-5	-4	-4	-2	-2	-2
-6	-6	-6	-6	-5	-3	-3
-6	-6	-6	-6	-5	-4	-4
-6	-6	-6	-6	-5	-5	-5
-6	-6	-6	-6	-6	-6	-6

→

-5	-4	-3	-2	-1	0
-5	-4	-3	-2	-1	-1
-5	-4	-4	-2	-2	-2
-5	-5	-5	-5	-3	-3
-5	-5	-5	-5	-4	-4
-5	-5	-5	-5	-5	-5

→

-4	-3	-2	-1	0
-4	-3	-2	-1	-1
-4	-4	-2	-2	-2
-4	-4	-4	-3	-3
-4	-4	-4	-4	-4

→

-3	-2	-1	0
-3	-2	-1	-1
-3	-2	-2	-2
-3	-3	-3	-3

→

-2	-1	0
-2	-1	-1
-2	-2	-2

→

-2	0
-2	-2

→

0

Chain of one-element Rees quotients

-7	-6	-5	-4	-3	-2	-1	0
-7	-7	-5	-4	-3	-2	-1	-1
-7	-7	-5	-4	-4	-2	-2	-2
-7	-7	-7	-7	-7	-5	-3	-3
-7	-7	-7	-7	-7	-5	-4	-4
-7	-7	-7	-7	-7	-5	-5	-5
-7	-7	-7	-7	-7	-7	-7	-6
-7	-7	-7	-7	-7	-7	-7	-7

-6	-5	-4	-3	-2	-1	0
-6	-5	-4	-3	-2	-1	-1
-6	-5	-4	-4	-2	-2	-2
-6	-6	-6	-6	-5	-3	-3
-6	-6	-6	-6	-5	-4	-4
-6	-6	-6	-6	-5	-5	-5
-6	-6	-6	-6	-6	-6	-6

-5	-4	-3	-2	-1	0
-5	-4	-3	-2	-1	-1
-5	-4	-4	-2	-2	-2
-5	-5	-5	-5	-3	-3
-5	-5	-5	-5	-4	-4
-5	-5	-5	-5	-5	-5

-4	-3	-2	-1	0
-4	-3	-2	-1	-1
-4	-4	-2	-2	-2
-4	-4	-4	-3	-3
-4	-4	-4	-4	-4

-3	-2	-1	0
-3	-2	-1	-1
-3	-2	-2	-2
-3	-3	-3	-3

-2	-1	0
-2	-1	-1
-2	-2	-2

-2	0
-2	-2

0

Rees coextensions

- $\bar{S}$ ... finite negative tomonoid
- S ... Rees quotient of $\bar{S}$ w.r.t. given $q \in \bar{S}$

$$S = \bar{S}/_{\approx_q}$$

- $\bar{S}$ is called *Rees coextension* of S
- if $q = \alpha$ (atom of $\bar{S}$)

  $\Rightarrow$ $\bar{S}$ is called *one-element Rees coextension* of S

While Rees quotients are unique, Rees coextensions are not.

Rees coextensions

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Tree of one-element Rees coextensions

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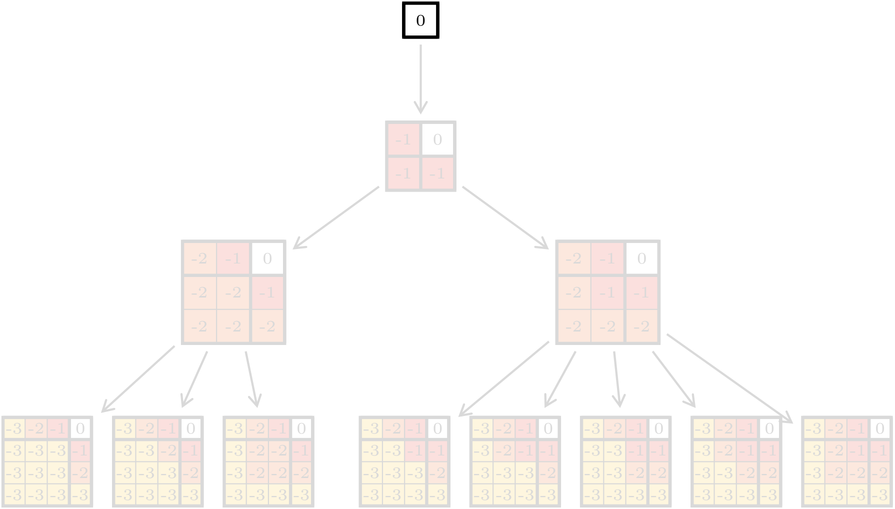
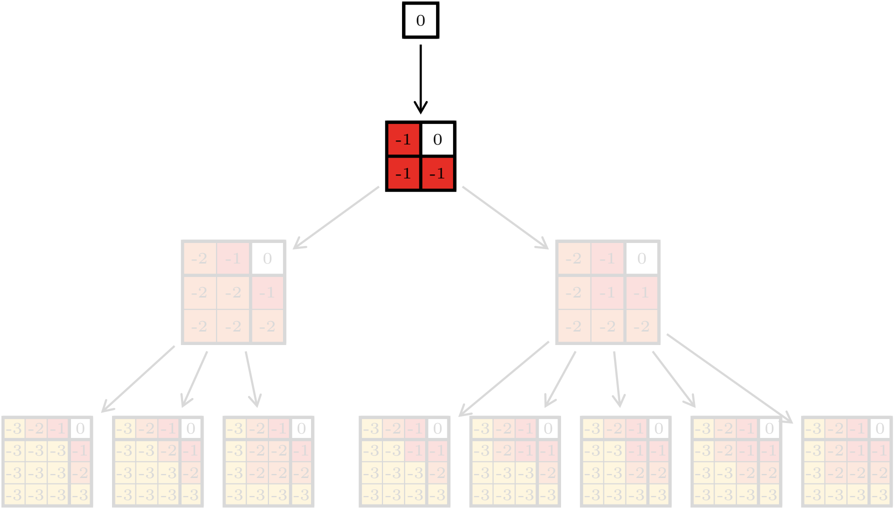
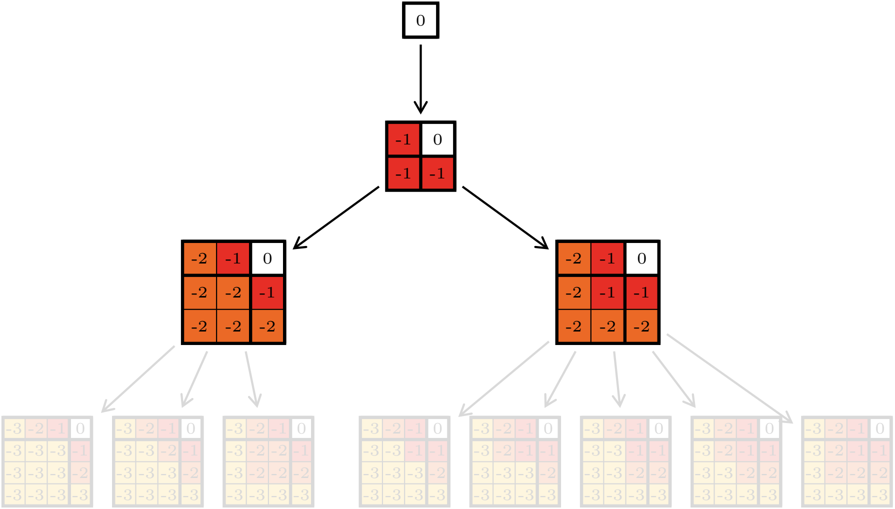


Tree of one-element Rees coextensions



Tree of one-element Rees coextensions



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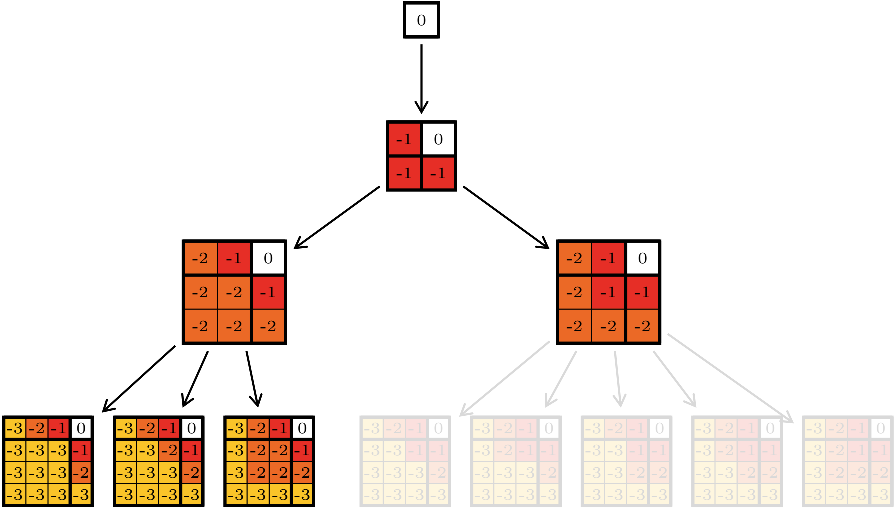

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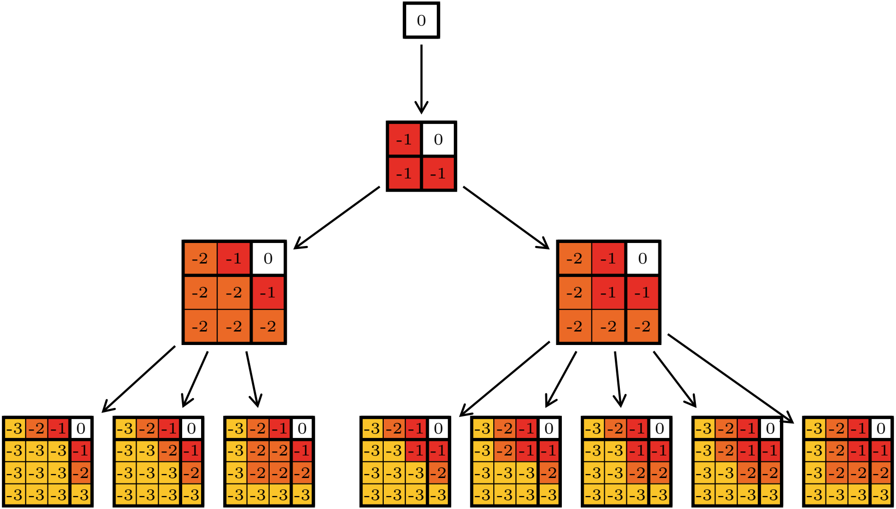

Question

How to determine all the one-element Rees coextensions of a given finite, negative tomonoid?

Construction of one-element co-extensions

$$\big(S^2;\, \trianglelefteq,\, \sim,\, (0,0)\big) \qquad \longrightarrow \qquad \big(\bar{S}^2;\, \trianglelefteq,\, \dot{\sim},\, (0,0)\big)$$

tomonoid partition → one-element co-extensions

S ... starting tomonoid

$S^\star$ ... S with bottom element removed

$\bar{S} = S^\star \,\dot{\cup}\, \{\bot, \alpha\}$, such that $\bot < \alpha < x$ for every $x \in S^\star$

... *bottom-doubling extension* of S

Construction of one-element co-extensions

$\big(\bar{S}^2;\, \trianglelefteq,\, \dot{\sim},\, (0{,}0)\big)$ is a one-element co-extension of $\big(S^2;\, \trianglelefteq,\, \sim,\, (0{,}0)\big)$ if:

(E1) $\forall\, (a,b),(c,d) \in \mathcal{P}\colon \quad (a,b) \sim (c,d) \quad \Rightarrow \quad (a,b) \,\dot{\sim}\, (c,d)$

(E2) $\forall\, (a,b),(b,c) \in \mathcal{P}\colon \quad (a,b) \sim d \;\text{ and }\; (b,c) \sim e \;\Rightarrow\; (d,c) \,\dot{\sim}\, (a,e)$

(E3) (a) $\forall\, (a,b) \in \mathcal{Q}\colon \quad (-1,a) \sim d \quad \Rightarrow \quad (d,b) \,\dot{\sim}\, \perp$
(b) $\forall\, (a,b) \in \mathcal{Q}\colon \quad (b,-1) \sim e \quad \Rightarrow \quad (a,e) \,\dot{\sim}\, \perp$

(E4) (a) $\forall\, b < 0\colon \quad (\perp,0) \,\dot{\sim}\, (0,\perp) \,\dot{\sim}\, (\alpha,b) \,\dot{\sim}\, (b,\alpha)$
(b) $(\alpha,0) \,\dot{\sim}\, (0,\alpha)$
(c) $\forall\, (a,b),(c,d) \in \mathcal{Q}\colon \quad (a,b) \trianglelefteq (c,d) \,\dot{\sim}\, \perp \quad \Rightarrow \quad (a,b) \,\dot{\sim}\, \perp$

$$\mathcal{P} = \big\{(a,b) \in S^2 \mid \text{there is } c \in S^\star \text{ such that } (a,b) \sim c\big\}$$
$$\mathcal{Q} = \bar{S}^2 \setminus \mathcal{P}$$

Example: Construction of a one-element co-extension

$$\big(S^2;\, \trianglelefteq,\, \sim,\, (0{,}0)\big) \;\longrightarrow\; \big(\bar{S}^2;\, \trianglelefteq,\, \dot{\sim},\, (0{,}0)\big)$$

Step: Bottom-doubling extension

- S … starting tomonoid
- $S^\star$ … bottom element removed
- $\bar{S}$ … we add $\perp$ and α s.t. $\perp < \alpha < x$ for every $x \in S^\star$

Question

- Which pairs shall be assigned to $\perp$ and which to α?

S

-5	-4	-3	-2	-1	0
-5	-5	-5	-4	-3	-1
-5	-5	-5	-5	-5	-2
-5	-5	-5	-5	-5	-3
-5	-5	-5	-5	-5	-4
-5	-5	-5	-5	-5	-5

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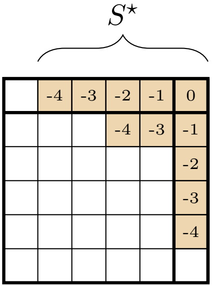

Example: Construction of a one-element co-extension

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Step: Bottom-doubling extension

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- $S^\star$ ... bottom element removed
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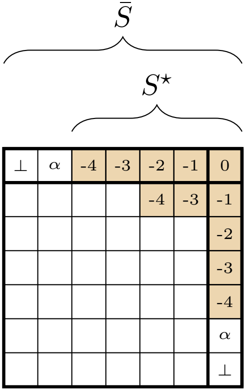

Question

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Example: Construction of a one-element co-extension

$$\big(S^2;\, \trianglelefteq,\, \sim,\, (0{,}0)\big) \;\longrightarrow\; \big(\bar{S}^2;\, \trianglelefteq,\, \dot{\sim},\, (0{,}0)\big)$$

Step: Bottom-doubling extension

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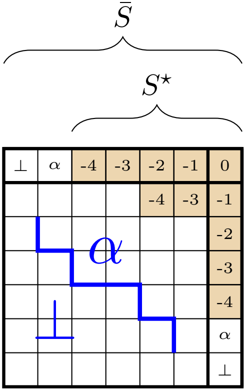

Question

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Example: Construction of a one-element co-extension

$$\big(S^2;\, \trianglelefteq,\, \sim,\, (0,0)\big) \;\longrightarrow\; \big(\bar{S}^2;\, \trianglelefteq,\, \dot{\sim},\, (0,0)\big)$$

Step (E1)

- $\forall\ (a,b),(c,d) \in \mathcal{P}:$

$$(a,b) \sim (c,d) \quad\Rightarrow\quad (a,b) \mathrel{\dot{\sim}} (c,d)$$

$\perp$	α	-4	-3	-2	-1	0
				-4	-3	-1
						-2
						-3
						-4
						α
						$\perp$

Example: Construction of a one-element co-extension

$$\big(S^2;\, \trianglelefteq,\, \sim,\, (0{,}0)\big) \;\longrightarrow\; \big(\bar{S}^2;\, \trianglelefteq,\, \dot{\sim},\, (0{,}0)\big)$$

Step (E1)

- $\forall\, (a,b), (c,d) \in \mathcal{P}$:

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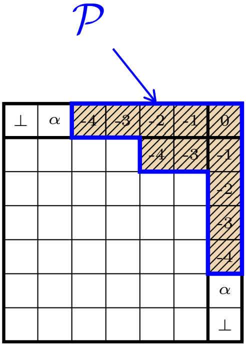


Example: Construction of a one-element co-extension

$$\big(S^2;\, \trianglelefteqslant,\, \sim,\, (0,0)\big) \;\longrightarrow\; \big(\bar{S}^2;\, \trianglelefteqslant,\, \dot{\sim},\, (0,0)\big)$$

Step (E2)

- $\forall\, (a,b),(b,c) \in \mathcal{P}$:

$$(a,b) \sim d \quad \text{and} \quad (b,c) \sim e$$

$$\Rightarrow \quad (d,c) \,\dot{\sim}\, (a,e)$$

$\bot$	α	-4	-3	-2	-1	0
				-4	-3	-1
						-2
						-3
						-4
						α
						$\bot$

Example: Construction of a one-element co-extension

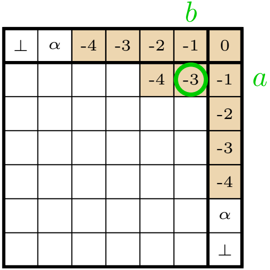
$$\big(S^2;\ \trianglelefteqslant,\ \sim,\ (0{,}0)\big) \ \longrightarrow\ \big(\bar{S}^2;\ \trianglelefteqslant,\ \dot{\sim},\ (0{,}0)\big)$$

Step (E2)

- $\forall\ (a,b), (b,c) \in \mathcal{P}:$

$$(a,b) \sim d \quad \text{and} \quad (b,c) \sim e$$

$$\Rightarrow \quad (d,c) \,\dot{\sim}\, (a,e)$$



Example: Construction of a one-element co-extension

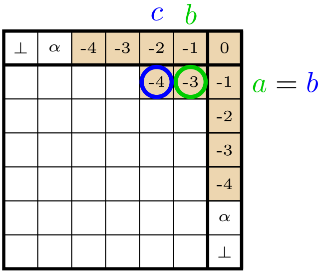
$$\big(S^2;\ \trianglelefteqslant,\ \sim,\ (0{,}0)\big)\ \longrightarrow\ \big(\bar{S}^2;\ \trianglelefteqslant,\ \dot{\sim},\ (0{,}0)\big)$$

Step (E2)

- $\forall\ (a,b), (b,c) \in \mathcal{P}:$

  $(a,b) \sim d \quad \text{and} \quad (b,c) \sim e$

  $\Rightarrow \quad (d,c) \mathrel{\dot{\sim}} (a,e)$



Example: Construction of a one-element co-extension

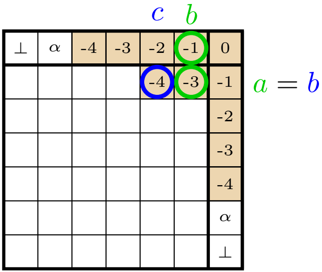
$$\big(S^2;\ \trianglelefteqslant,\ \sim,\ (0,0)\big) \ \longrightarrow\ \big(\bar{S}^2;\ \trianglelefteqslant,\ \dot{\sim},\ (0,0)\big)$$

Step (E2)

- $\forall\ (a,b), (b,c) \in \mathcal{P}:$

  $(a,b) \sim d \quad \text{and} \quad (b,c) \sim e$

  $$\Rightarrow \quad (d,c) \,\dot{\sim}\, (a,e)$$



Example: Construction of a one-element co-extension

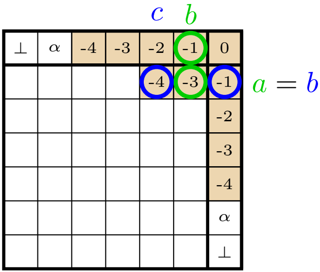
$$\big(S^2;\, \trianglelefteq,\, \sim,\, (0{,}0)\big) \;\longrightarrow\; \big(\bar{S}^2;\, \trianglelefteq,\, \dot{\sim},\, (0{,}0)\big)$$

Step (E2)

- $\forall\ (a, b), (b, c) \in \mathcal{P}$:

  $$(a, b) \sim d \quad \text{and} \quad (b, c) \sim e$$

  $$\Rightarrow \quad (d, c) \;\dot{\sim}\; (a, e)$$



Example: Construction of a one-element co-extension

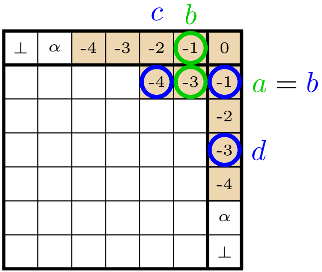
$$\big(S^2;\ \trianglelefteq,\ \sim,\ (0{,}0)\big) \ \longrightarrow\ \big(\bar{S}^2;\ \trianglelefteq,\ \dot{\sim},\ (0{,}0)\big)$$

Step (E2)

- $\forall\ (a,b), (b,c) \in \mathcal{P}:$

$$(a,b) \sim d \quad \text{and} \quad (b,c) \sim e$$

$$\Rightarrow \quad (d,c) \mathbin{\dot{\sim}} (a,e)$$



Example: Construction of a one-element co-extension

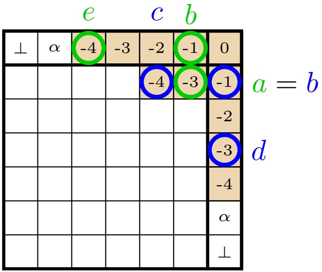
$$\big(S^2;\, \trianglelefteq,\, \sim,\, (0{,}0)\big) \;\longrightarrow\; \big(\bar{S}^2;\, \trianglelefteq,\, \dot{\sim},\, (0{,}0)\big)$$

Step (E2)

- $\forall\ (a,b), (b,c) \in \mathcal{P}$:

  $(a,b) \sim d$ and $(b,c) \sim e$

  $\Rightarrow \quad (d,c)\ \dot{\sim}\ (a,e)$



Example: Construction of a one-element co-extension

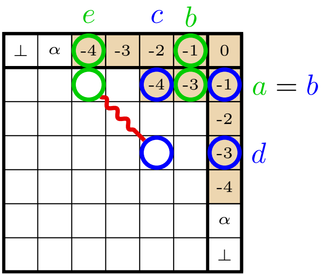
$$\big(S^2;\, \trianglelefteq,\, \sim,\, (0{,}0)\big) \;\longrightarrow\; \big(\bar{S}^2;\, \trianglelefteq,\, \dot{\sim},\, (0{,}0)\big)$$

Step (E2)

- $\forall\ (a,b), (b,c) \in \mathcal{P}$:

$$(a,b) \sim d \quad \text{and} \quad (b,c) \sim e$$

$$\Rightarrow \quad (d,c) \mathrel{\dot{\sim}} (a,e)$$



Example: Construction of a one-element co-extension

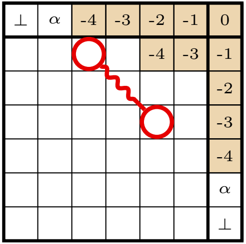
$$\big(S^2;\, \trianglelefteq,\, \sim,\, (0,0)\big) \;\longrightarrow\; \big(\bar{S}^2;\, \trianglelefteq,\, \dot{\sim},\, (0,0)\big)$$

Step (E2)

- $\forall\ (a,b), (b,c) \in \mathcal{P}$:

$$(a,b) \sim d \quad \text{and} \quad (b,c) \sim e$$

$$\Rightarrow \quad (d,c) \,\dot{\sim}\, (a,e)$$



Example: Construction of a one-element co-extension

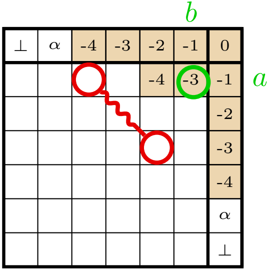
$$\big(S^2;\, \trianglelefteq,\, \sim,\, (0{,}0)\big) \;\longrightarrow\; \big(\bar{S}^2;\, \trianglelefteq,\, \dot{\sim},\, (0{,}0)\big)$$

Step (E2)

- $\forall\ (a,b), (b,c) \in \mathcal{P}:$

$$(a,b) \sim d \quad \text{and} \quad (b,c) \sim e$$

$$\Rightarrow \quad (d,c) \mathrel{\dot{\sim}} (a,e)$$



Example: Construction of a one-element co-extension

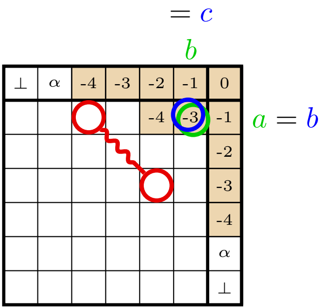
$$\big(S^2;\, \trianglelefteq,\, \sim,\, (0{,}0)\big) \;\longrightarrow\; \big(\bar{S}^2;\, \trianglelefteq,\, \dot{\sim},\, (0{,}0)\big)$$

Step (E2)

- $\forall\ (a,b), (b,c) \in \mathcal{P}:$

$$(a,b) \sim d \quad \text{and} \quad (b,c) \sim e$$

$$\Rightarrow \quad (d,c) \mathrel{\dot{\sim}} (a,e)$$



Example: Construction of a one-element co-extension

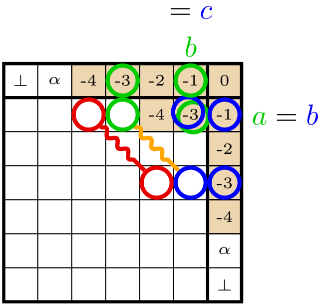
$$\big(S^2;\ \trianglelefteq,\ \sim,\ (0{,}0)\big) \ \longrightarrow\ \big(\bar{S}^2;\ \trianglelefteq,\ \dot{\sim},\ (0{,}0)\big)$$

Step (E2)

- $\forall\ (a,b), (b,c) \in \mathcal{P}$:

$$(a,b) \sim d \quad \text{and} \quad (b,c) \sim e$$

$$\Rightarrow \quad (d,c)\ \dot{\sim}\ (a,e)$$



Example: Construction of a one-element co-extension

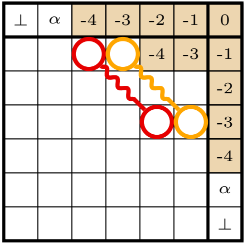
$$\big(S^2;\, \trianglelefteqslant,\, \sim,\, (0{,}0)\big) \;\longrightarrow\; \big(\bar{S}^2;\, \trianglelefteqslant,\, \dot{\sim},\, (0{,}0)\big)$$

Step (E2)

- $\forall\ (a,b), (b,c) \in \mathcal{P}$:

$$(a,b) \sim d \quad \text{and} \quad (b,c) \sim e$$

$$\Rightarrow \quad (d,c) \mathrel{\dot{\sim}} (a,e)$$

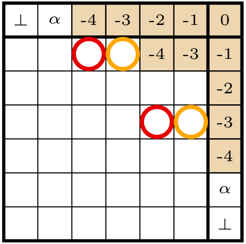


Example: Construction of a one-element co-extension

$$\big(S^2;\ \trianglelefteq,\ \sim,\ (0{,}0)\big) \;\longrightarrow\; \big(\bar{S}^2;\ \trianglelefteq,\ \dot{\sim},\ (0{,}0)\big)$$

Step (E3)

- $\forall\ (a,b) \in \mathcal{Q}$:
  - $(-1,a) \sim d \quad \Rightarrow \quad (d,b) \mathrel{\dot{\sim}} \bot$
  - $(b,-1) \sim e \quad \Rightarrow \quad (a,e) \mathrel{\dot{\sim}} \bot$

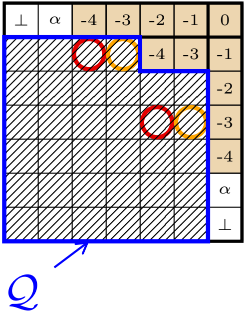


Example: Construction of a one-element co-extension

$$\big(S^2;\, \trianglelefteq,\, \sim,\, (0,0)\big) \;\longrightarrow\; \big(\bar{S}^2;\, \trianglelefteq,\, \dot{\sim},\, (0,0)\big)$$

Step (E3)

- $\forall\, (a,b) \in \mathcal{Q}$:
  - $(-1,a) \sim d \quad \Rightarrow \quad (d,b) \mathrel{\dot{\sim}} \bot$
  - $(b,-1) \sim e \quad \Rightarrow \quad (a,e) \mathrel{\dot{\sim}} \bot$

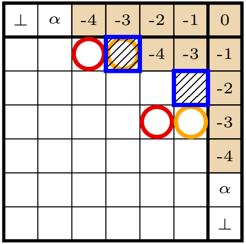


Example: Construction of a one-element co-extension

$$\big(S^2;\, \trianglelefteq,\, \sim,\, (0{,}0)\big) \;\longrightarrow\; \big(\bar{S}^2;\, \trianglelefteq,\, \dot{\sim},\, (0{,}0)\big)$$

Step (E3)

- $\forall\, (a,b) \in \mathcal{Q}$:
  - $(-1,a) \sim d \quad \Rightarrow \quad (d,b) \mathrel{\dot{\sim}} \bot$
  - $(b,-1) \sim e \quad \Rightarrow \quad (a,e) \mathrel{\dot{\sim}} \bot$

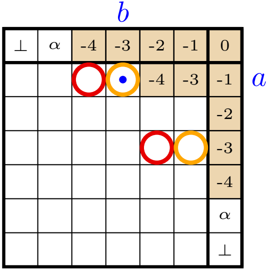


Example: Construction of a one-element co-extension

$$\big(S^2;\, \trianglelefteq,\, \sim,\, (0{,}0)\big) \;\longrightarrow\; \big(\bar{S}^2;\, \trianglelefteq,\, \dot{\sim},\, (0{,}0)\big)$$

Step (E3)

- $\forall\ (a, b) \in \mathcal{Q}$:
  - $(-1, a) \sim d \quad \Rightarrow \quad (d, b) \mathrel{\dot{\sim}} \bot$
  - $(b, -1) \sim e \quad \Rightarrow \quad (a, e) \mathrel{\dot{\sim}} \bot$

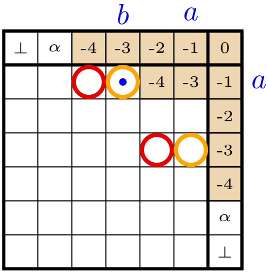


Example: Construction of a one-element co-extension

$$\big(S^2;\, \trianglelefteq,\, \sim,\, (0{,}0)\big) \;\longrightarrow\; \big(\bar{S}^2;\, \trianglelefteq,\, \dot{\sim},\, (0{,}0)\big)$$

Step (E3)

- $\forall\ (a, b) \in \mathcal{Q}$:
  - $(-1, a) \sim d \quad \Rightarrow \quad (d, b) \,\dot{\sim}\, \bot$
  - $(b, -1) \sim e \quad \Rightarrow \quad (a, e) \,\dot{\sim}\, \bot$

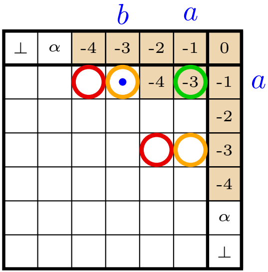


Example: Construction of a one-element co-extension

$$\big(S^2;\ \trianglelefteq,\ \sim,\ (0{,}0)\big) \ \longrightarrow\ \big(\bar{S}^2;\ \trianglelefteq,\ \dot{\sim},\ (0{,}0)\big)$$

Step (E3)

- $\forall\ (a, b) \in \mathcal{Q}$:
  - $(-1, a) \sim d \quad \Rightarrow \quad (d, b)\ \dot{\sim}\ \bot$
  - $(b, -1) \sim e \quad \Rightarrow \quad (a, e)\ \dot{\sim}\ \bot$

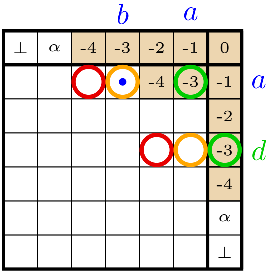


Example: Construction of a one-element co-extension

$$\big(S^2;\, \trianglelefteq,\, \sim,\, (0{,}0)\big) \;\longrightarrow\; \big(\bar{S}^2;\, \trianglelefteq,\, \dot{\sim},\, (0{,}0)\big)$$

Step (E3)

- $\forall\ (a,b) \in \mathcal{Q}$:
  - $(-1,a) \sim d \quad \Rightarrow \quad (d,b) \,\dot{\sim}\, \bot$
  - $(b,-1) \sim e \quad \Rightarrow \quad (a,e) \,\dot{\sim}\, \bot$

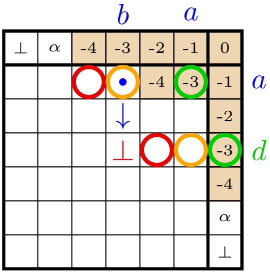


Example: Construction of a one-element co-extension

$$\big(S^2;\ \trianglelefteq,\ \sim,\ (0,0)\big) \ \longrightarrow\ \big(\bar{S}^2;\ \trianglelefteq,\ \dot{\sim},\ (0,0)\big)$$

Step (E3)

- $\forall\ (a,b) \in \mathcal{Q}$:
  - $(-1,a) \sim d \quad \Rightarrow \quad (d,b)\ \dot{\sim}\ \bot$
  - $(b,-1) \sim e \quad \Rightarrow \quad (a,e)\ \dot{\sim}\ \bot$

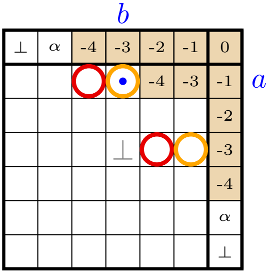


Example: Construction of a one-element co-extension

$$\big(S^2;\, \trianglelefteq,\, \sim,\, (0{,}0)\big) \;\longrightarrow\; \big(\bar{S}^2;\, \trianglelefteq,\, \dot{\sim},\, (0{,}0)\big)$$

Step (E3)

- $\forall\ (a,b) \in \mathcal{Q}$:
  - $(-1,a) \sim d \quad\Rightarrow\quad (d,b)\ \dot{\sim}\ \bot$
  - $(b,-1) \sim e \quad\Rightarrow\quad (a,e)\ \dot{\sim}\ \bot$

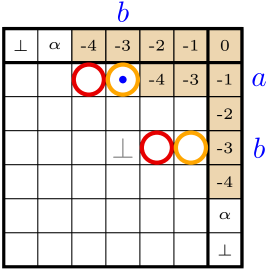


Example: Construction of a one-element co-extension

$$\big(S^2;\, \trianglelefteq,\, \sim,\, (0,0)\big) \;\longrightarrow\; \big(\bar{S}^2;\, \trianglelefteq,\, \dot{\sim},\, (0,0)\big)$$

Step (E3)

- $\forall\, (a,b) \in \mathcal{Q}$:
  - $(-1,a) \sim d \quad \Rightarrow \quad (d,b) \mathrel{\dot{\sim}} \bot$
  - $(b,-1) \sim e \quad \Rightarrow \quad (a,e) \mathrel{\dot{\sim}} \bot$

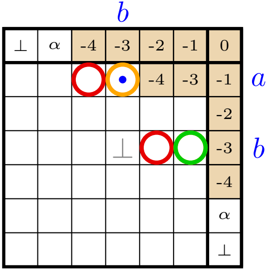


Example: Construction of a one-element co-extension

$$\big(S^2;\ \trianglelefteq,\ \sim,\ (0{,}0)\big) \;\longrightarrow\; \big(\bar{S}^2;\ \trianglelefteq,\ \dot{\sim},\ (0{,}0)\big)$$

Step (E3)

- $\forall\ (a, b) \in \mathcal{Q}$:
  - $(-1, a) \sim d \quad \Rightarrow \quad (d, b)\ \dot{\sim}\ \bot$
  - $(b, -1) \sim e \quad \Rightarrow \quad (a, e)\ \dot{\sim}\ \bot$

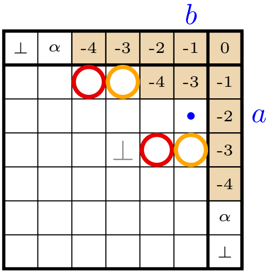


Example: Construction of a one-element co-extension

$$\big(S^2;\, \trianglelefteq,\, \sim,\, (0,0)\big) \;\longrightarrow\; \big(\bar{S}^2;\, \trianglelefteq,\, \dot{\sim},\, (0,0)\big)$$

Step (E3)

- $\forall\; (a,b) \in \mathcal{Q}$:
  - $(-1,a) \sim d \quad \Rightarrow \quad (d,b) \;\dot{\sim}\; \bot$
  - $(b,-1) \sim e \quad \Rightarrow \quad (a,e) \;\dot{\sim}\; \bot$

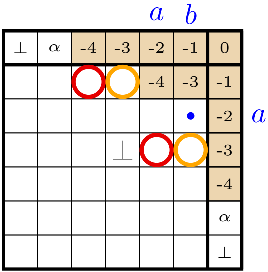


Example: Construction of a one-element co-extension

$$\big(S^2;\ \trianglelefteq,\ \sim,\ (0,0)\big) \ \longrightarrow\ \big(\bar{S}^2;\ \trianglelefteq,\ \dot{\sim},\ (0,0)\big)$$

Step (E3)

- $\forall\ (a, b) \in \mathcal{Q}$:
  - $(-1, a) \sim d \quad \Rightarrow \quad (d, b) \mathrel{\dot{\sim}} \bot$
  - $(b, -1) \sim e \quad \Rightarrow \quad (a, e) \mathrel{\dot{\sim}} \bot$

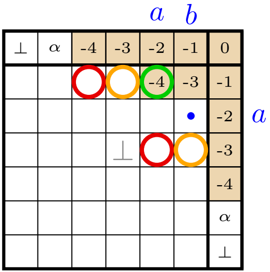


Example: Construction of a one-element co-extension

$$\big(S^2;\ \trianglelefteqslant,\ \sim,\ (0{,}0)\big) \ \longrightarrow\ \big(\bar{S}^2;\ \trianglelefteqslant,\ \dot{\sim},\ (0{,}0)\big)$$

Step (E3)

- $\forall\ (a,b) \in \mathcal{Q}$:
  - $(-1,a) \sim d \quad \Rightarrow \quad (d,b)\ \dot{\sim}\ \bot$
  - $(b,-1) \sim e \quad \Rightarrow \quad (a,e)\ \dot{\sim}\ \bot$

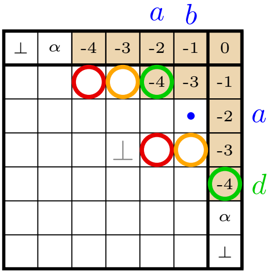


Example: Construction of a one-element co-extension

$$\big(S^2;\, \trianglelefteq,\, \sim,\, (0{,}0)\big) \;\longrightarrow\; \big(\bar{S}^2;\, \trianglelefteq,\, \dot{\sim},\, (0{,}0)\big)$$

Step (E3)

- $\forall\ (a, b) \in \mathcal{Q}$:
  - $(-1, a) \sim d \quad \Rightarrow \quad (d, b) \mathrel{\dot\sim} \bot$
  - $(b, -1) \sim e \quad \Rightarrow \quad (a, e) \mathrel{\dot\sim} \bot$

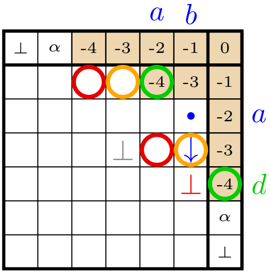


Example: Construction of a one-element co-extension

$$\big(S^2;\, \trianglelefteqslant,\, \sim,\, (0{,}0)\big) \;\longrightarrow\; \big(\bar{S}^2;\, \trianglelefteqslant,\, \dot{\sim},\, (0{,}0)\big)$$

Step (E3)

- $\forall\ (a,b) \in \mathcal{Q}$:
  - $(-1,a) \sim d \quad \Rightarrow \quad (d,b)\ \dot{\sim}\ \bot$
  - $(b,-1) \sim e \quad \Rightarrow \quad (a,e)\ \dot{\sim}\ \bot$

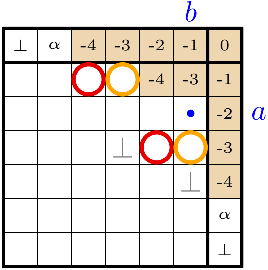


Example: Construction of a one-element co-extension

$$\big(S^2;\ \trianglelefteq,\ \sim,\ (0,0)\big) \ \longrightarrow\ \big(\bar{S}^2;\ \trianglelefteq,\ \dot{\sim},\ (0,0)\big)$$

Step (E3)

- $\forall\ (a, b) \in \mathcal{Q}$:
  - $(-1, a) \sim d \quad \Rightarrow \quad (d, b) \mathrel{\dot{\sim}} \bot$
  - $(b, -1) \sim e \quad \Rightarrow \quad (a, e) \mathrel{\dot{\sim}} \bot$

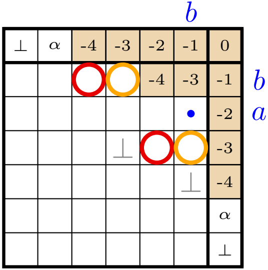


Example: Construction of a one-element co-extension

$$\big(S^2;\ \trianglelefteq,\ \sim,\ (0{,}0)\big) \ \longrightarrow\ \big(\bar{S}^2;\ \trianglelefteq,\ \dot{\sim},\ (0{,}0)\big)$$

Step (E3)

- $\forall\ (a, b) \in \mathcal{Q}$:
  - $(-1, a) \sim d \quad \Rightarrow \quad (d, b)\ \dot{\sim}\ \bot$
  - $(b, -1) \sim e \quad \Rightarrow \quad (a, e)\ \dot{\sim}\ \bot$

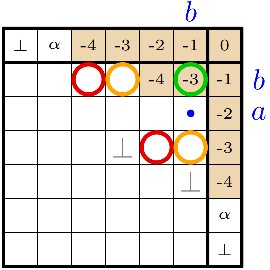


Example: Construction of a one-element co-extension

$$\big(S^2;\, \trianglelefteq,\, \sim,\, (0{,}0)\big) \;\longrightarrow\; \big(\bar S^2;\, \trianglelefteq,\, \dot\sim,\, (0{,}0)\big)$$

Step (E3)

- $\forall\ (a,b) \in \mathcal{Q}$:
  - $(-1,a) \sim d \quad \Rightarrow \quad (d,b)\ \dot\sim\ \bot$
  - $(b,-1) \sim e \quad \Rightarrow \quad (a,e)\ \dot\sim\ \bot$

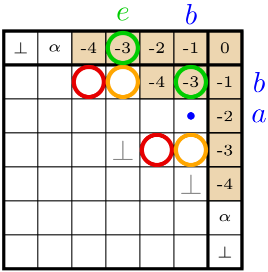


Example: Construction of a one-element co-extension

$$\big(S^2;\ \trianglelefteq,\ \sim,\ (0{,}0)\big) \ \longrightarrow\ \big(\bar{S}^2;\ \trianglelefteq,\ \dot{\sim},\ (0{,}0)\big)$$

Step (E3)

- $\forall\ (a, b) \in \mathcal{Q}$:
  - $(-1, a) \sim d \quad \Rightarrow \quad (d, b) \mathrel{\dot{\sim}} \bot$
  - $(b, -1) \sim e \quad \Rightarrow \quad (a, e) \mathrel{\dot{\sim}} \bot$

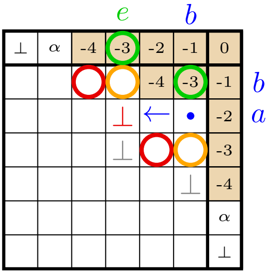


Example: Construction of a one-element co-extension

$$\big(S^2;\, \trianglelefteq,\, \sim,\, (0{,}0)\big) \;\longrightarrow\; \big(\bar{S}^2;\, \trianglelefteq,\, \dot{\sim},\, (0{,}0)\big)$$

Step (E3)

- $\forall\, (a, b) \in \mathcal{Q}$:
  - $(-1, a) \sim d \quad \Rightarrow \quad (d, b) \mathrel{\dot{\sim}} \bot$
  - $(b, -1) \sim e \quad \Rightarrow \quad (a, e) \mathrel{\dot{\sim}} \bot$

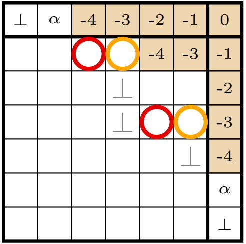


Example: Construction of a one-element co-extension

$$\big(S^2;\, \trianglelefteq,\, \sim,\, (0{,}0)\big) \;\longrightarrow\; \big(\bar{S}^2;\, \trianglelefteq,\, \dot{\sim},\, (0{,}0)\big)$$

Step (E3)

- $\forall\ (a,b) \in \mathcal{Q}$:
  - $(-1,a) \sim d \quad \Rightarrow \quad (d,b) \mathrel{\dot{\sim}} \bot$
  - $(b,-1) \sim e \quad \Rightarrow \quad (a,e) \mathrel{\dot{\sim}} \bot$



Example: Construction of a one-element co-extension

$$\big(S^2;\ \trianglelefteq,\ \sim,\ (0{,}0)\big)\ \longrightarrow\ \big(\bar{S}^2;\ \trianglelefteq,\ \dot{\sim},\ (0{,}0)\big)$$

Step (E4)

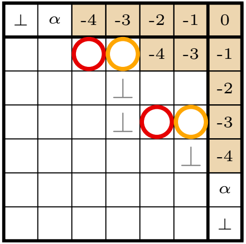
(a) $\forall\, b < 0$:

$$(\bot, 0)\ \dot{\sim}\ (0, \bot)\ \dot{\sim}\ (\alpha, b)\ \dot{\sim}\ (b, \alpha)$$

(b) $(\alpha, 0)\ \dot{\sim}\ (0, \alpha)$

(c) $\forall\, (a, b), (c, d) \in \mathcal{Q}$:

$$(a, b) \trianglelefteq (c, d)\ \dot{\sim}\ \bot \quad \Rightarrow \quad (a, b)\ \dot{\sim}\ \bot$$



Example: Construction of a one-element co-extension

$$\big(S^2;\, \trianglelefteq,\, \sim,\, (0{,}0)\big) \;\longrightarrow\; \big(\bar{S}^2;\, \trianglelefteq,\, \dot{\sim},\, (0{,}0)\big)$$

Step (E4)

(a) $\forall\, b < 0$:

$$(\bot, 0) \,\dot{\sim}\, (0, \bot) \,\dot{\sim}\, (\alpha, b) \,\dot{\sim}\, (b, \alpha)$$

(b) $(\alpha, 0) \,\dot{\sim}\, (0, \alpha)$

(c) $\forall\, (a, b), (c, d) \in \mathcal{Q}$:

$$(a, b) \trianglelefteq (c, d) \,\dot{\sim}\, \bot \quad \Rightarrow \quad (a, b) \,\dot{\sim}\, \bot$$

$\bot$	α	-4	-3	-2	-1	0
$\bot$	$\bot$			-4	-3	-1
$\bot$	$\bot$		$\bot$			-2
$\bot$	$\bot$		$\bot$			-3
$\bot$	$\bot$				$\bot$	-4
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	α
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$

Example: Construction of a one-element co-extension

$$\big(S^2;\, \trianglelefteq,\, \sim,\, (0,0)\big) \;\longrightarrow\; \big(\bar{S}^2;\, \trianglelefteq,\, \dot{\sim},\, (0,0)\big)$$

Step (E4)

(a) $\forall\, b < 0$:

$$(\perp, 0) \,\dot{\sim}\, (0, \perp) \,\dot{\sim}\, (\alpha, b) \,\dot{\sim}\, (b, \alpha)$$

(b) $(\alpha, 0) \,\dot{\sim}\, (0, \alpha)$

(c) $\forall\, (a, b), (c, d) \in \mathcal{Q}$:

$$(a, b) \trianglelefteq (c, d) \,\dot{\sim}\, \perp \quad \Rightarrow \quad (a, b) \,\dot{\sim}\, \perp$$

$\perp$	α	-4	-3	-2	-1	0
$\perp$	$\perp$			-4	-3	-1
$\perp$	$\perp$		$\perp$			-2
$\perp$	$\perp$		$\perp$			-3
$\perp$	$\perp$				$\perp$	-4
$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	α
$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$

Example: Construction of a one-element co-extension

$$\big(S^2;\ \trianglelefteq,\ \sim,\ (0{,}0)\big) \ \longrightarrow\ \big(\bar{S}^2;\ \trianglelefteq,\ \dot{\sim},\ (0{,}0)\big)$$

Step (E4)

(a) $\forall\, b < 0:$

$$(\bot, 0) \mathrel{\dot\sim} (0, \bot) \mathrel{\dot\sim} (\alpha, b) \mathrel{\dot\sim} (b, \alpha)$$

(b) $(\alpha, 0) \mathrel{\dot\sim} (0, \alpha)$

(c) $\forall\, (a, b), (c, d) \in \mathcal{Q}:$

$$(a, b) \trianglelefteq (c, d) \mathrel{\dot\sim} \bot \quad \Rightarrow \quad (a, b) \mathrel{\dot\sim} \bot$$

$\bot$	α	-4	-3	-2	-1	0
$\bot$	$\bot$	○	○	-4	-3	-1
$\bot$	$\bot$	$\bot$	$\bot$			-2
$\bot$	$\bot$	$\bot$	$\bot$	○	○	-3
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	-4
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	α
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$

Example: Construction of a one-element co-extension

$$\big(S^2;\ \trianglelefteq,\ \sim,\ (0{,}0)\big) \ \longrightarrow\ \big(\bar{S}^2;\ \trianglelefteq,\ \dot\sim,\ (0{,}0)\big)$$

(E1) $\forall\, (a,b),(c,d) \in \mathcal{P}$:

$$(a,b) \sim (c,d) \quad \Rightarrow \quad (a,b) \mathrel{\dot\sim} (c,d)$$

(E2) $\forall\, (a,b),(b,c) \in \mathcal{P}$:

$$(a,b) \sim d \quad \text{and} \quad (b,c) \sim e$$
$$\Rightarrow \quad (d,c) \mathrel{\dot\sim} (a,e)$$

(E3) $\forall\, (a,b) \in \mathcal{Q}$:

- $(-1,a) \sim d \quad \Rightarrow \quad (d,b) \mathrel{\dot\sim} \bot$
- $(b,-1) \sim e \quad \Rightarrow \quad (a,e) \mathrel{\dot\sim} \bot$

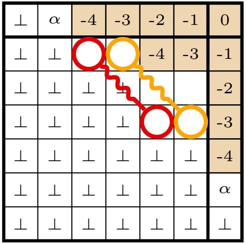
(E4) (a) $\forall\, b < 0$:

$$(\bot,0) \mathrel{\dot\sim} (0,\bot) \mathrel{\dot\sim} (\alpha,b) \mathrel{\dot\sim} (b,\alpha)$$

(b) $(\alpha,0) \mathrel{\dot\sim} (0,\alpha)$

(c) $\forall\, (a,b),(c,d) \in \mathcal{Q}$:

$$(a,b) \trianglelefteq (c,d) \mathrel{\dot\sim} \bot \quad \Rightarrow \quad (a,b) \mathrel{\dot\sim} \bot$$



Example: Construction of a one-element co-extension

⊥	α	-4	-3	-2	-1	0
⊥	⊥	⊥	⊥	-4	-3	-1
⊥	⊥	⊥	⊥	⊥	⊥	-2
⊥	⊥	⊥	⊥	⊥	⊥	-3
⊥	⊥	⊥	⊥	⊥	⊥	-4
⊥	⊥	⊥	⊥	⊥	⊥	α
⊥	⊥	⊥	⊥	⊥	⊥	⊥

⊥	α	-4	-3	-2	-1	0
⊥	⊥	⊥	⊥	-4	-3	-1
⊥	⊥	⊥	⊥	⊥	α	-2
⊥	⊥	⊥	⊥	⊥	⊥	-3
⊥	⊥	⊥	⊥	⊥	⊥	-4
⊥	⊥	⊥	⊥	⊥	⊥	α
⊥	⊥	⊥	⊥	⊥	⊥	⊥

⊥	α	-4	-3	-2	-1	0
⊥	⊥	⊥	⊥	-4	-3	-1
⊥	⊥	⊥	⊥	α	α	-2
⊥	⊥	⊥	⊥	⊥	⊥	-3
⊥	⊥	⊥	⊥	⊥	⊥	-4
⊥	⊥	⊥	⊥	⊥	⊥	α
⊥	⊥	⊥	⊥	⊥	⊥	⊥

⊥	α	-4	-3	-2	-1	0
⊥	⊥	⊥	α	-4	-3	-1
⊥	⊥	⊥	⊥	⊥	α	-2
⊥	⊥	⊥	⊥	⊥	α	-3
⊥	⊥	⊥	⊥	⊥	⊥	-4
⊥	⊥	⊥	⊥	⊥	⊥	α
⊥	⊥	⊥	⊥	⊥	⊥	⊥

⊥	α	-4	-3	-2	-1	0
⊥	⊥	⊥	α	-4	-3	-1
⊥	⊥	⊥	⊥	α	α	-2
⊥	⊥	⊥	⊥	⊥	α	-3
⊥	⊥	⊥	⊥	⊥	⊥	-4
⊥	⊥	⊥	⊥	⊥	⊥	α
⊥	⊥	⊥	⊥	⊥	⊥	⊥

⊥	α	-4	-3	-2	-1	0
⊥	⊥	α	α	-4	-3	-1
⊥	⊥	⊥	⊥	α	α	-2
⊥	⊥	⊥	⊥	α	α	-3
⊥	⊥	⊥	⊥	⊥	⊥	-4
⊥	⊥	⊥	⊥	⊥	⊥	α
⊥	⊥	⊥	⊥	⊥	⊥	⊥

Concluding slide

aa

Thank you for your kind attention!