

On the structure of finite commutative totally ordered monoids

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Outline

- ▶ Introduction
 - ▶ Totally ordered monoids (Tomonoids)
- ▶ Tomonoid partitions
- ▶ Finite negative tomonoids
 - ▶ One-element Rees co-extensions
- ▶ General finite tomonoids

Totally ordered monoid

Monoid ... $(S; \oplus, 0)$

- S ... set
- \oplus ... associative binary operation on S
- 0 ... neutral (unit) element

Tomonoid (totally ordered monoid) ... $(S; \leqslant, \oplus, 0)$

- \leqslant ... total (linear) order on S that is *compatible* with \oplus :
 $a \leq b$ implies $a \oplus c \leq b \oplus c$ and $c \oplus a \leq c \oplus b$
for every $a, b, c \in S$

Totally ordered monoid

A tomonoid $(S; \leq, \oplus, 0)$ is called

finite ... if S is finite

negative ... if $0 = \top$

positive ... if $0 = \perp$

commutative ... if $a \oplus b = b \oplus a$ for every $a, b \in S$

Archimedean ... if \top , 0 , and \perp are the only idempotents

- $a \in S$ is *idempotent* if $a \oplus a = a$

Examples

$$(-8, -7, -6, -5, -4, -3, -2, -1, 0)$$

0	-8	-7	-6	-5	-4	-3	-2	-1	0
-1	-8	-8	-8	-5	-4	-3	-2	-1	-1
-2	-8	-8	-8	-5	-4	-4	-2	-2	-2
-3	-8	-8	-8	-8	-8	-5	-3	-3	-3
-4	-8	-8	-8	-8	-8	-5	-4	-4	-4
-5	-8	-8	-8	-8	-8	-5	-5	-5	-5
-6	-8	-8	-8	-8	-8	-8	-8	-6	-6
-7	-8	-8	-8	-8	-8	-8	-8	-7	-7
-8	-8	-8	-8	-8	-8	-8	-8	-8	-8
	-8	-7	-6	-5	-4	-3	-2	-1	0

$$(-2, -1, 0, 1, 2, 3, 4, 5, 6)$$

6	6	6	6	6	6	6	6	6	6
5	4	4	5	5	5	5	5	5	6
4	4	4	4	5	5	5	5	5	6
3	1	2	3	4	4	4	5	5	6
2	1	1	2	4	4	4	5	5	6
1	1	1	1	4	4	4	5	5	6
0	-2	-1	0	1	2	3	4	5	6
-1	-2	-2	-1	1	1	2	4	4	6
-2	-2	-2	-2	1	1	1	4	4	6
	-2	-1	0	1	2	3	4	5	6

- ▶ negative
- ▶ -1 and -2 are idempotents
- ▶ non-commutative
- ▶ non-Archimedean

- ▶ non-negative
- ▶ non-positive
- ▶ commutative
- ▶ Archimedean

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Level set representation of a tomonoid

- $(S; \leqslant, \oplus, 0)$... tomonoid
- level equivalence ... binary relation \sim on $S \times S = S^2$:
$$(a, b) \sim (c, d) \quad \text{if} \quad a \oplus b = c \oplus d$$

- $(S^2; \leqslant, \sim, (0,0))$... tomonoid partition of S

$$(a, b) \leqslant (c, d) \quad \text{if} \quad a \leq c \quad \text{and} \quad b \leq d$$

Example

0	-3	-2	-1	0
-1	-3	-3	-1	-1
-2	-3	-3	-2	-2
-3	-3	-3	-3	-3
	-3	-2	-1	0

equivalence classes of \sim

- $\{(0,0)\}$
- $\{(-1,0), (-1,-1), (0,-1)\}$
- $\{(-2,0), (-2,-1), (0,-2)\}$
- $\{(-3,0), (-3,-1), (-3,-2), (-3,-3), (-2,-3), (-1,-3), (0,-3), (-2,-2), (-1,-2)\}$

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Level set representation of a tomonoid

Level equivalence \sim has the following properties:

- (P1) For every $a, b, c, d, e \in S$, $(a, b) \sim (d, 0)$ and $(b, c) \sim (0, e)$ imply $(d, c) \sim (a, e)$.

► associativity

- (P2) For every $a, b \in S$ there is exactly one $c \in S$ such that $(a, b) \sim (0, c) \sim (c, 0)$.

► neutral element

- (P3) For every $a, b, c, d, a', b', c', d' \in S$,
 $(a, b) \sim (a', b') \leqslant (c, d) \sim (c', d') \leqslant (a, b)$ implies $(a, b) \sim (c, d)$.

► compatibility with the order

$(S^2; \leqslant, \sim, (0,0))$ satisfies (P1)–(P3) \Leftrightarrow it is a tomonoid partition

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Level set representation of a tomonoid

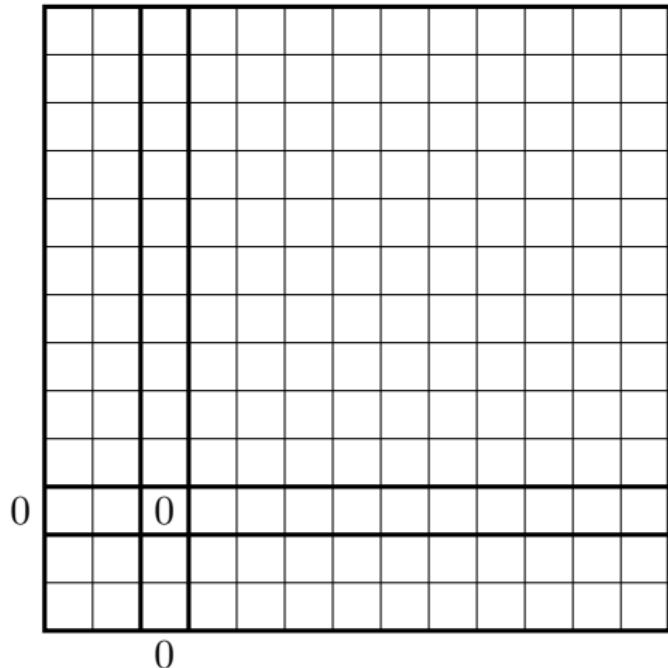
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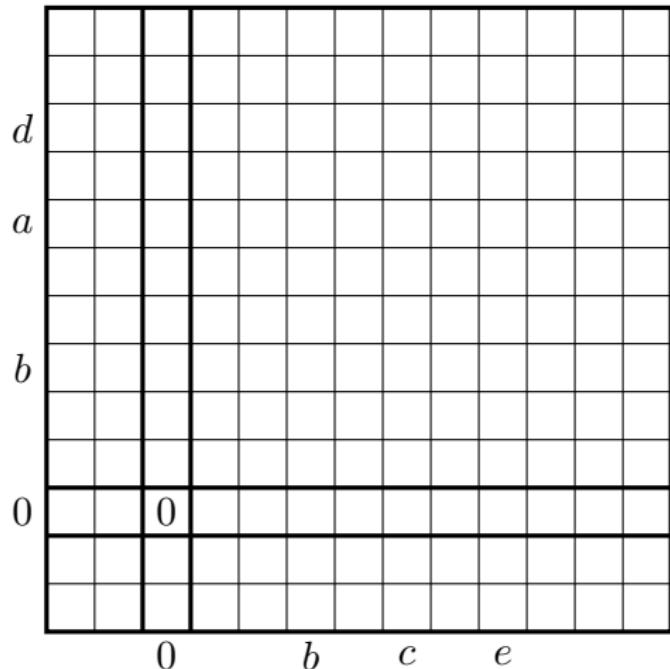
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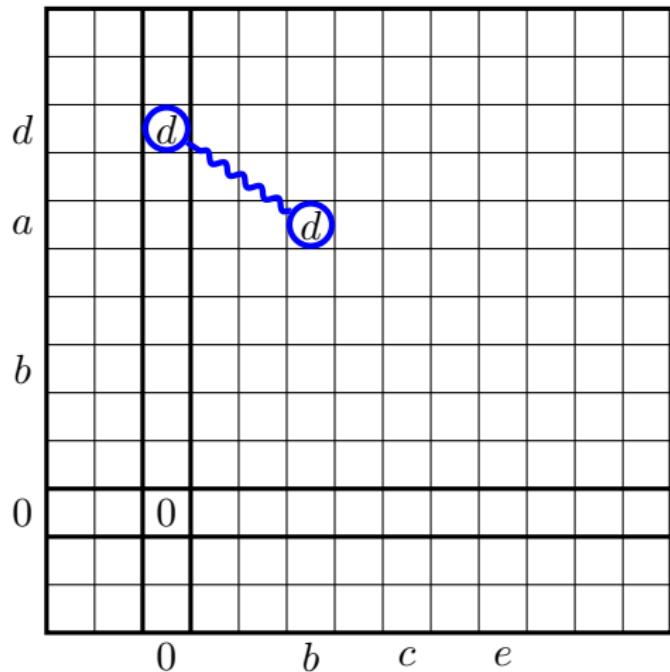
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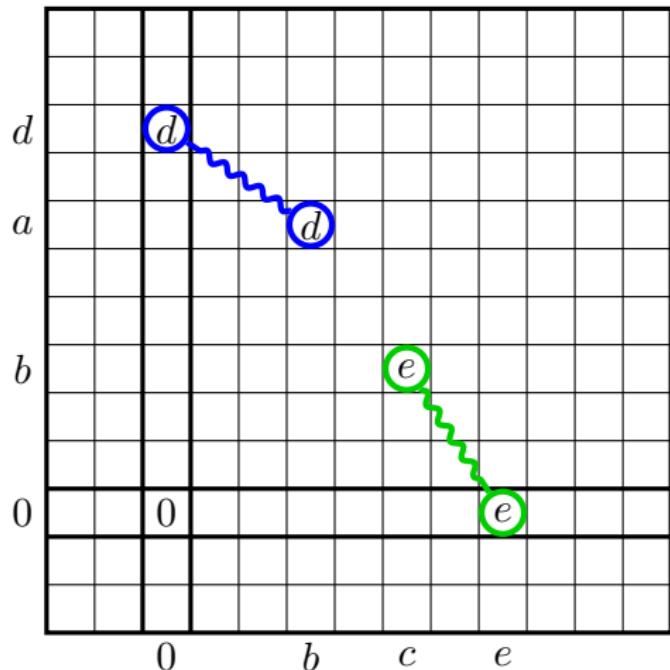
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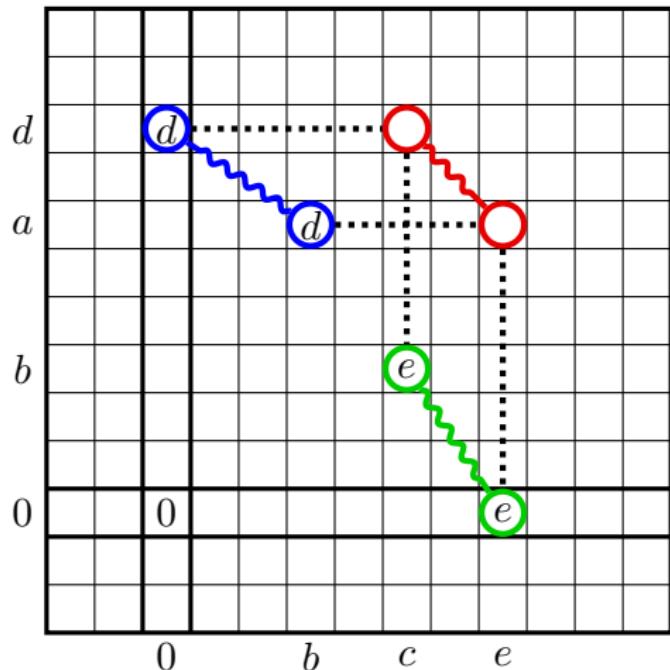
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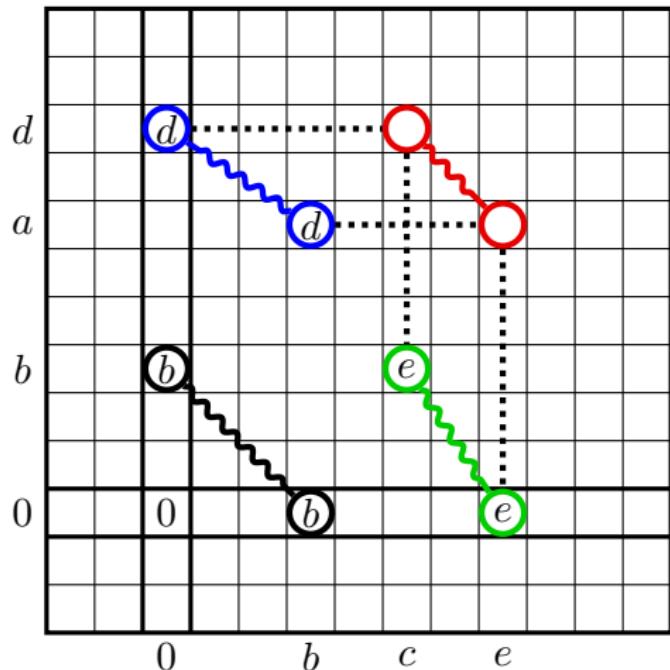
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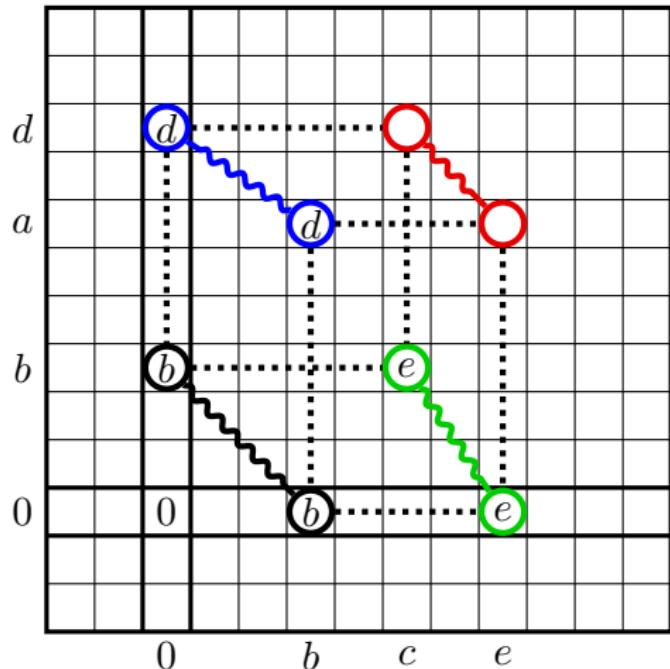
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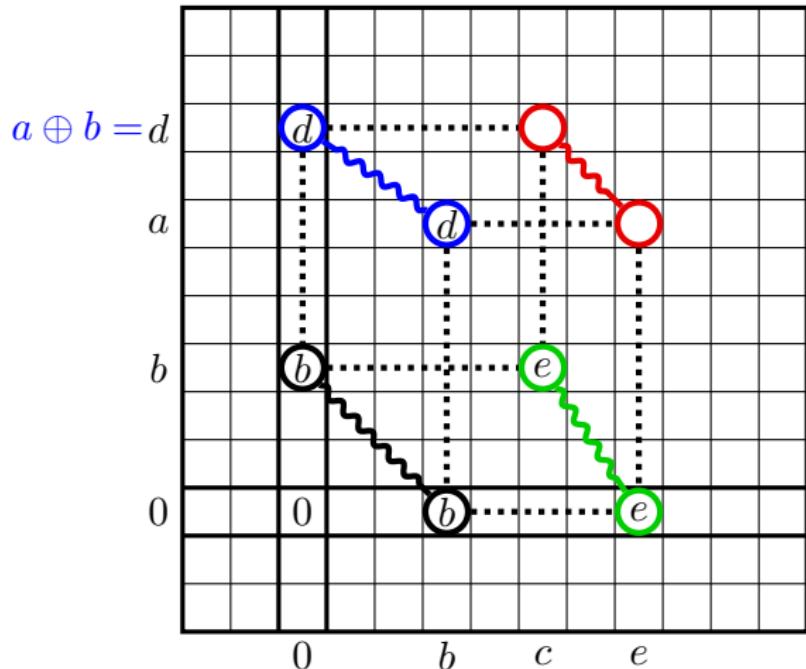
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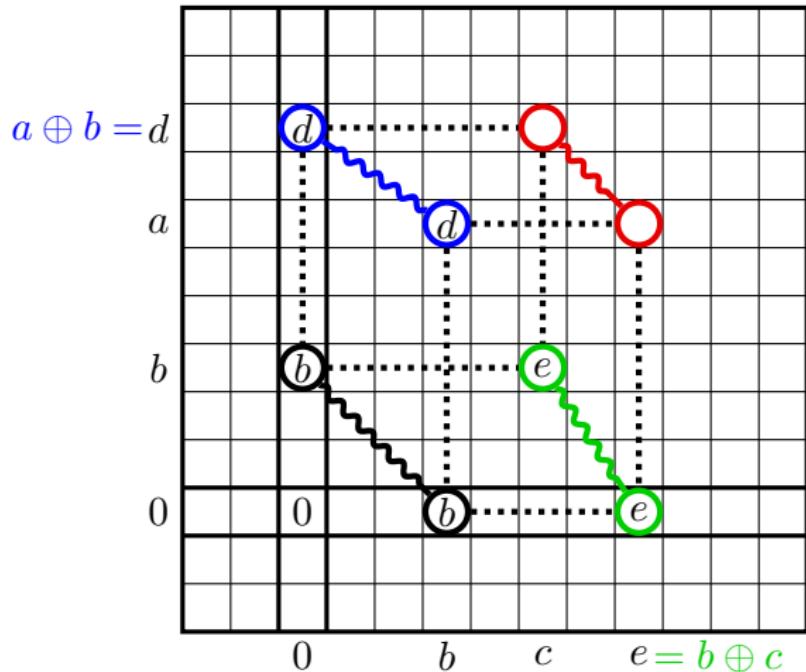
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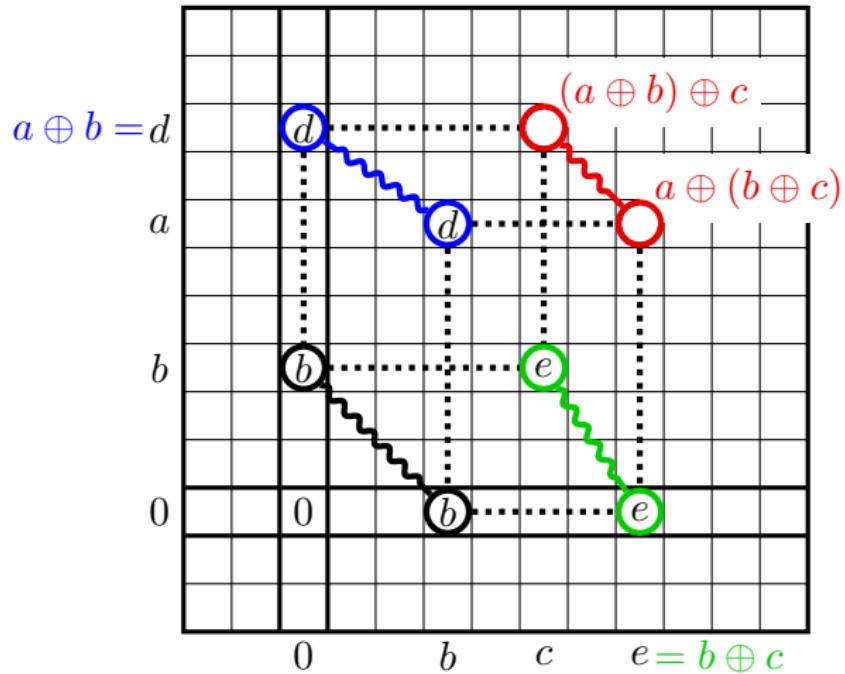
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Reidemeister condition

- ▶ Condition (P1) corresponds with the *Reidemeister closure condition* known from *web geometry* where it characterizes those loops that are associative (and thus they are groups).

-  W. Blaschke, G. Bol, *Geometrie der Gewebe, topologische Fragen der Differentialgeometrie*, Springer, 1939.
-  J. Aczél, *Quasigroups, nets and nomograms*, Advances in Mathematics, 1965.

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Rees quotients

- $(S; \leq, \oplus, 0)$... finite negative tomonoid
- for given $q \in S$ define binary relation \approx_q on S by

$$x \approx_q y \quad \text{if} \quad x = y \quad \text{or} \quad x, y \leq q$$

- \approx_q ... *Rees congruence* (with respect to q)
- S/\approx_q ... *Rees quotient* of S (with respect to q)

- if $q = \alpha$ (atom of S)

⇒ S/\approx_α is called *one-element Rees quotient* of S

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$$S: \quad 0 < u < v < w < x < y < z < 1$$

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$$\begin{array}{ccccccccc} & & & & & & q \\ S: & 0 & < u & < v & < w & < \textcolor{blue}{x} & < y & < z & < 1 \\ & \overbrace{\hspace{10em}} & & & & & & & \\ S/\approx_q: & \underline{0} & & & & & & & y < z < 1 \end{array}$$

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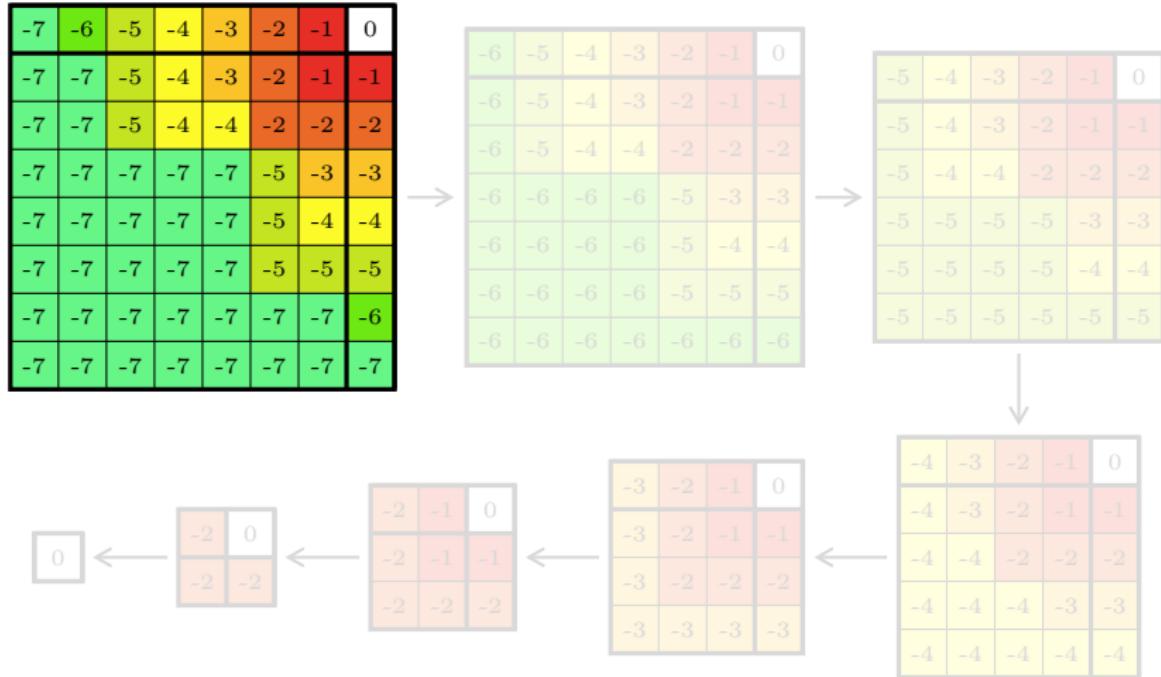
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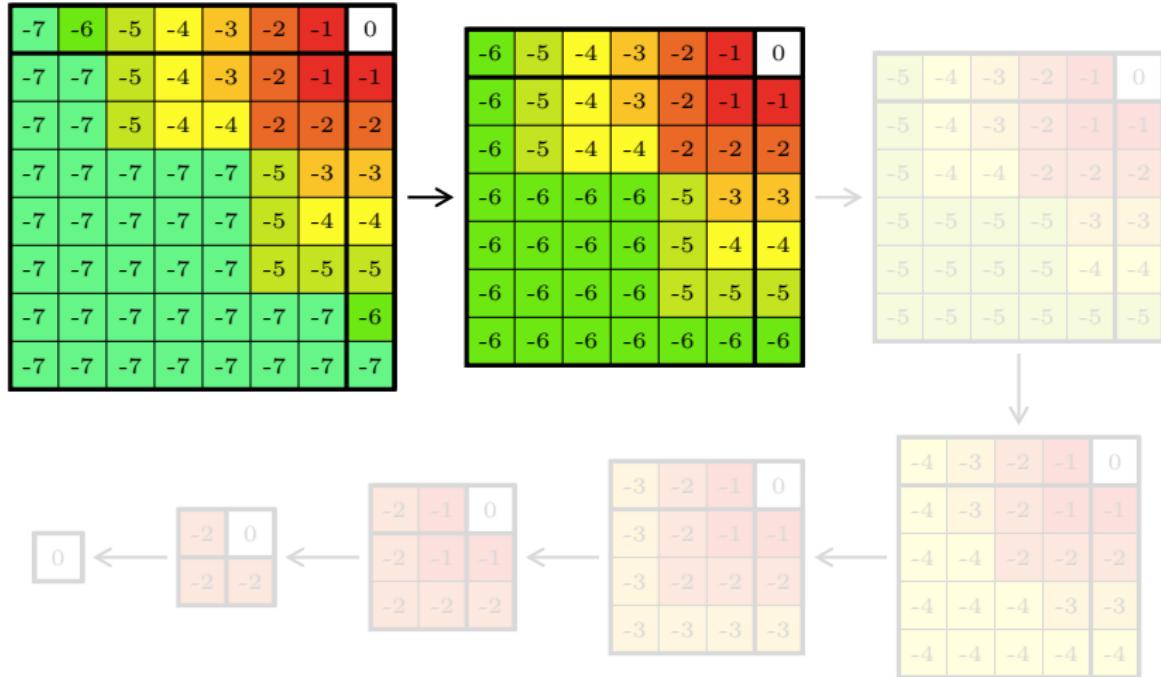
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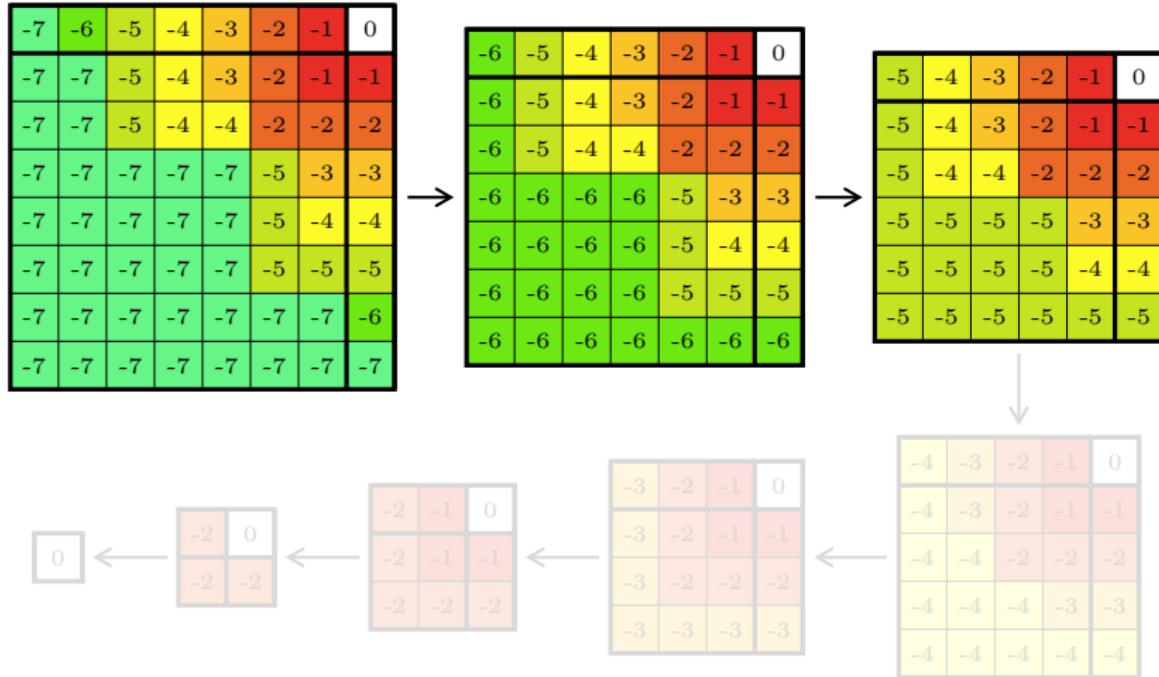
Chain of one-element Rees quotients



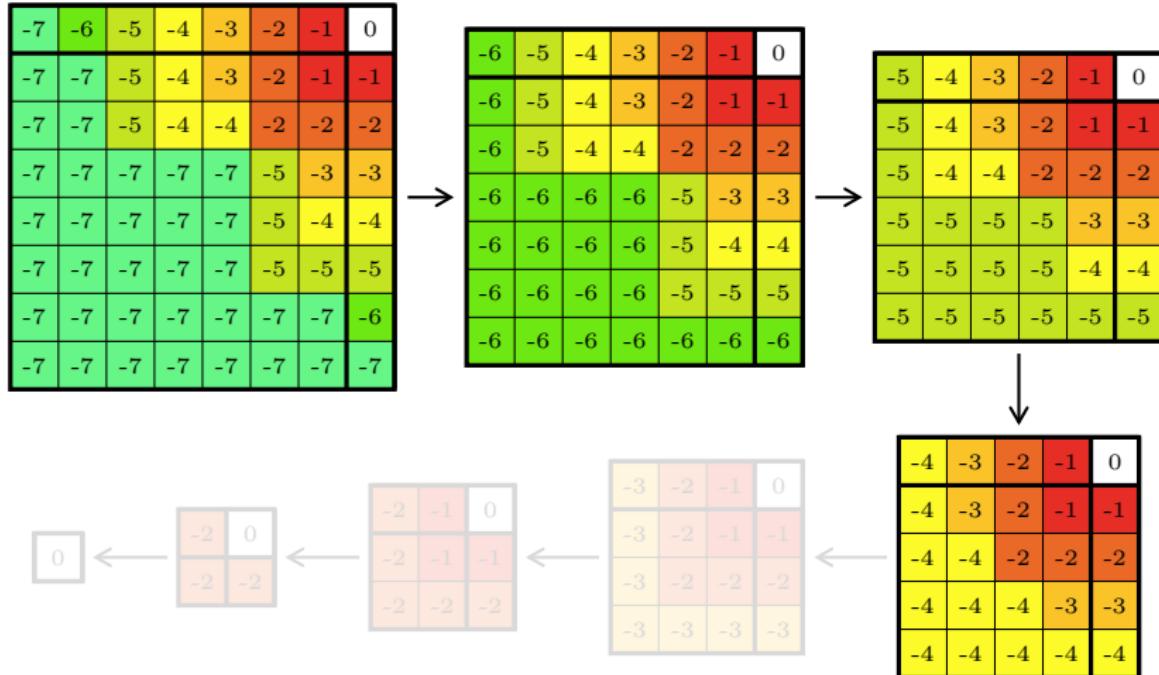
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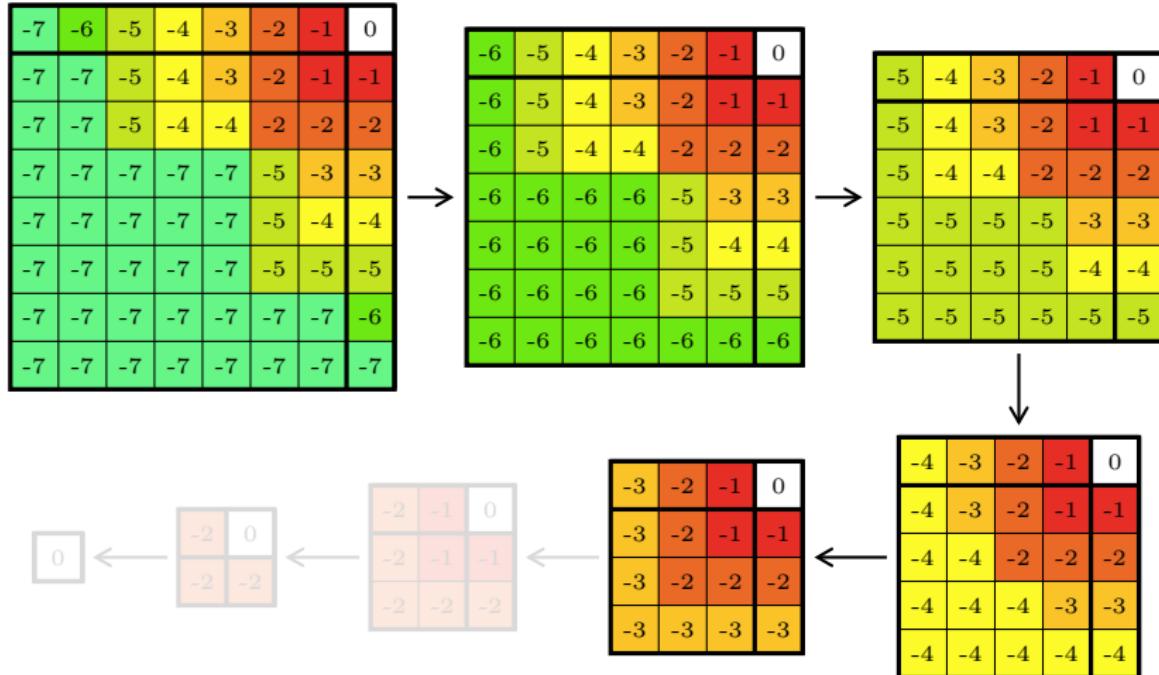
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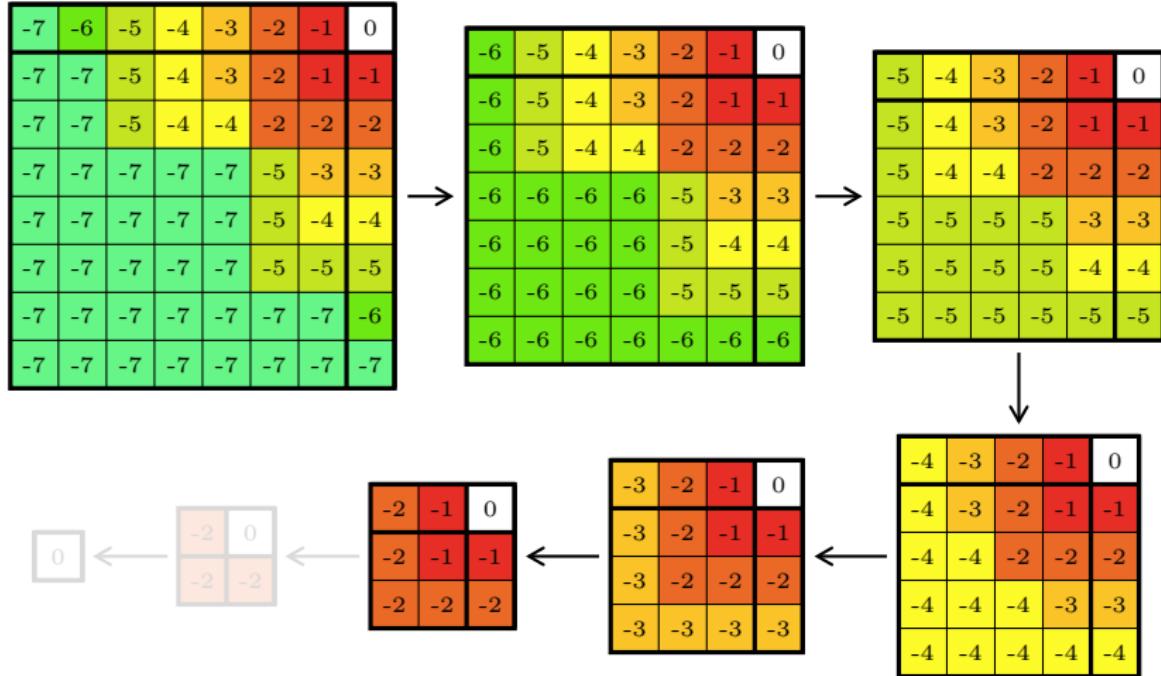
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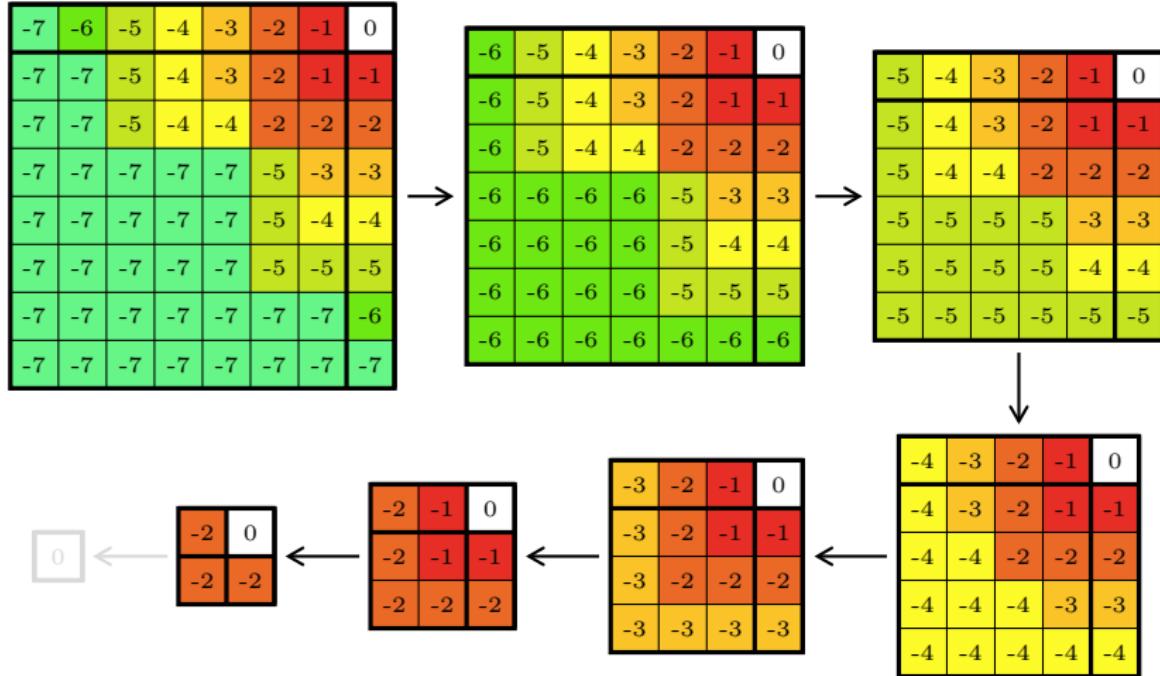
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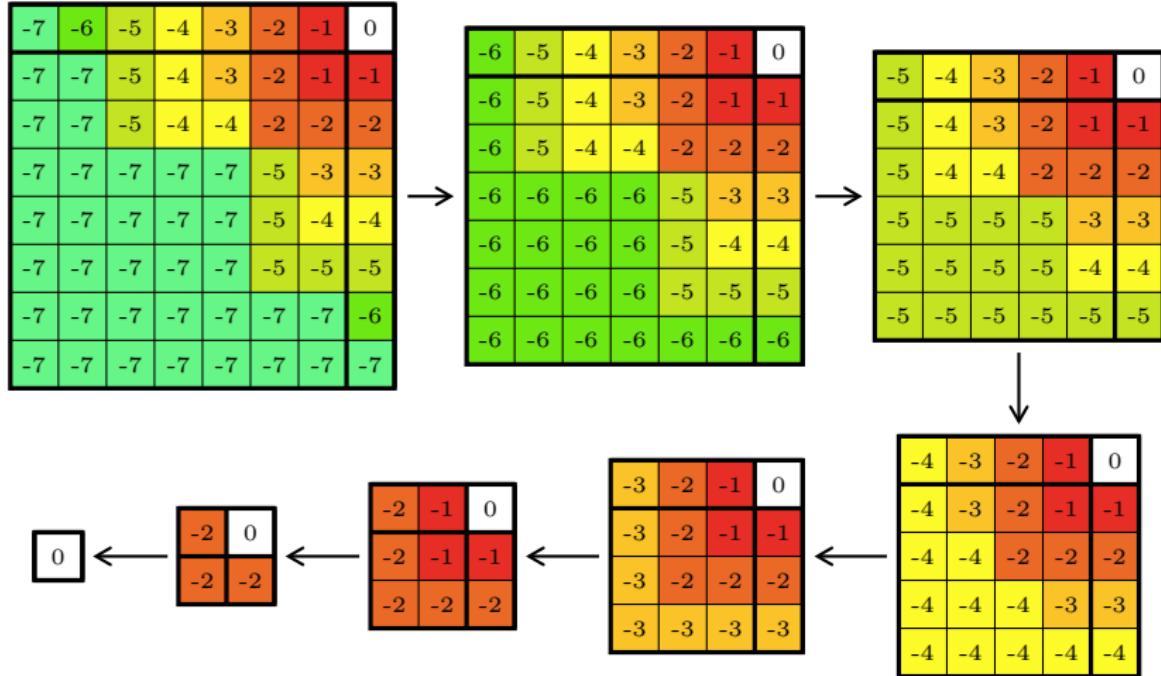
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Rees coextensions

- ▶ \bar{S} ... finite negative tomonoid
- ▶ S ... Rees quotient of \bar{S} w.r.t. given $q \in \bar{S}$

$$S = \bar{S}/\approx_q$$

- ▶ \bar{S} is called *Rees coextension* of S
- ▶ if $q = \alpha$ (atom of \bar{S})
⇒ \bar{S} is called *one-element Rees coextension* of S

While Rees quotients are unique, Rees coextensions are not.

Rees coextensions

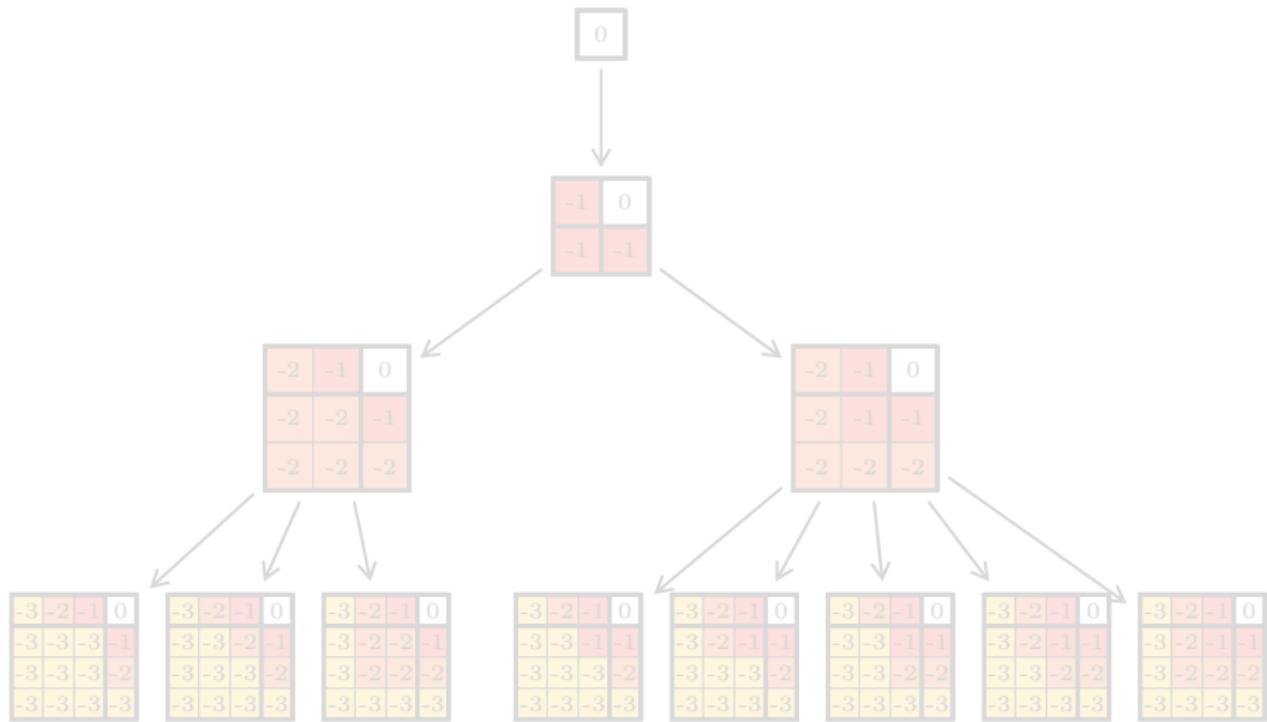
- ▶ \bar{S} ... finite negative tomonoid
- ▶ S ... Rees quotient of \bar{S} w.r.t. given $q \in \bar{S}$

$$S = \bar{S}/\approx_q$$

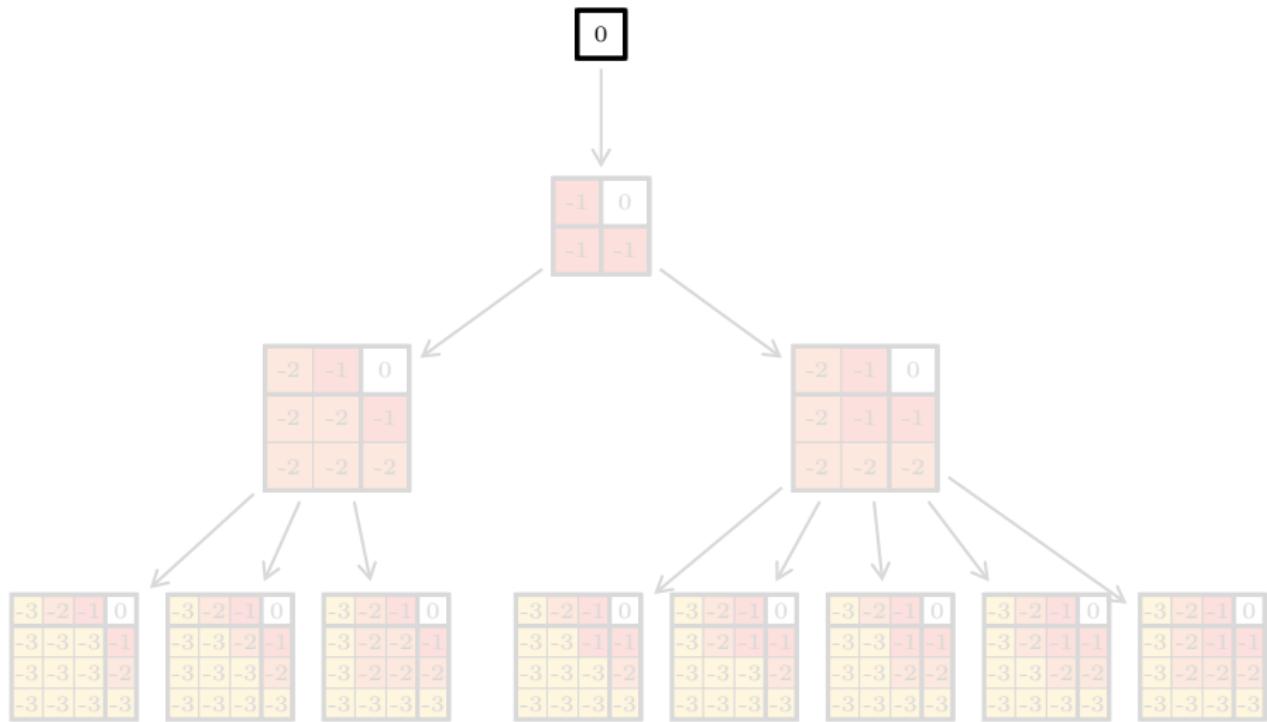
- ▶ \bar{S} is called *Rees coextension* of S
- ▶ if $q = \alpha$ (atom of \bar{S})
⇒ \bar{S} is called *one-element Rees coextension* of S

While Rees quotients are unique, Rees coextensions **are not**.

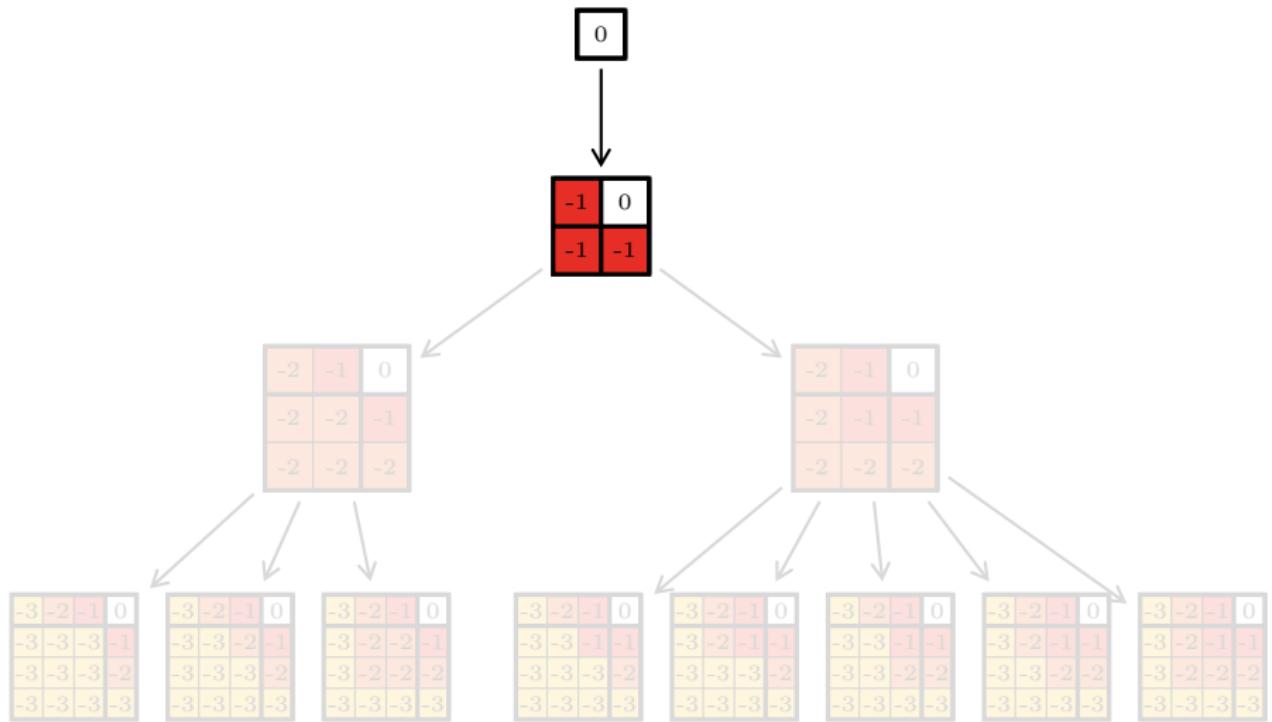
Tree of one-element Rees coextensions



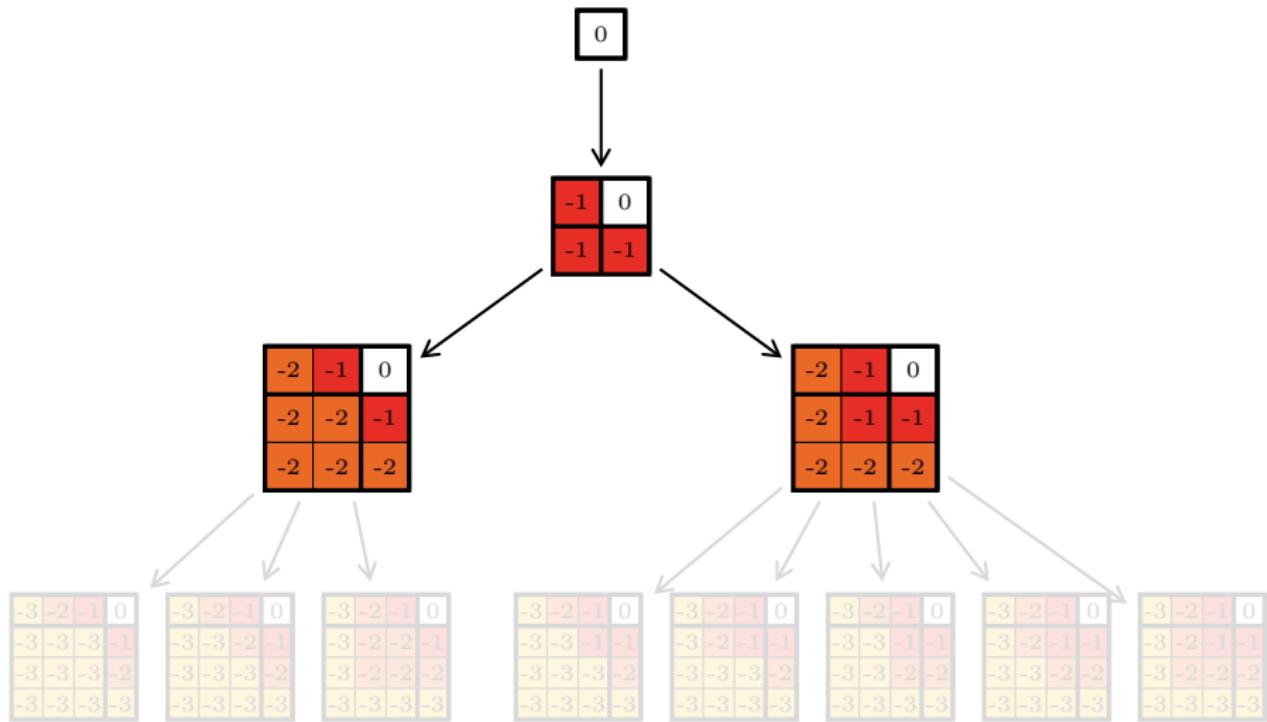
Tree of one-element Rees coextensions



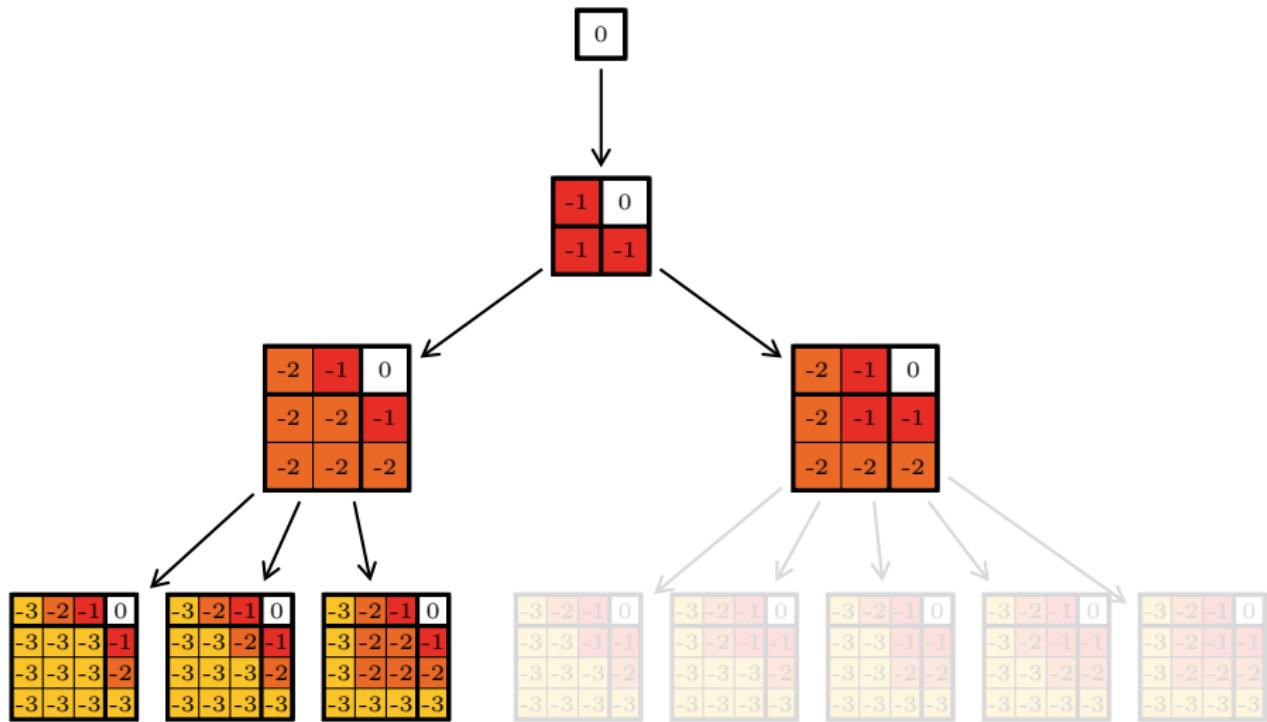
Tree of one-element Rees coextensions



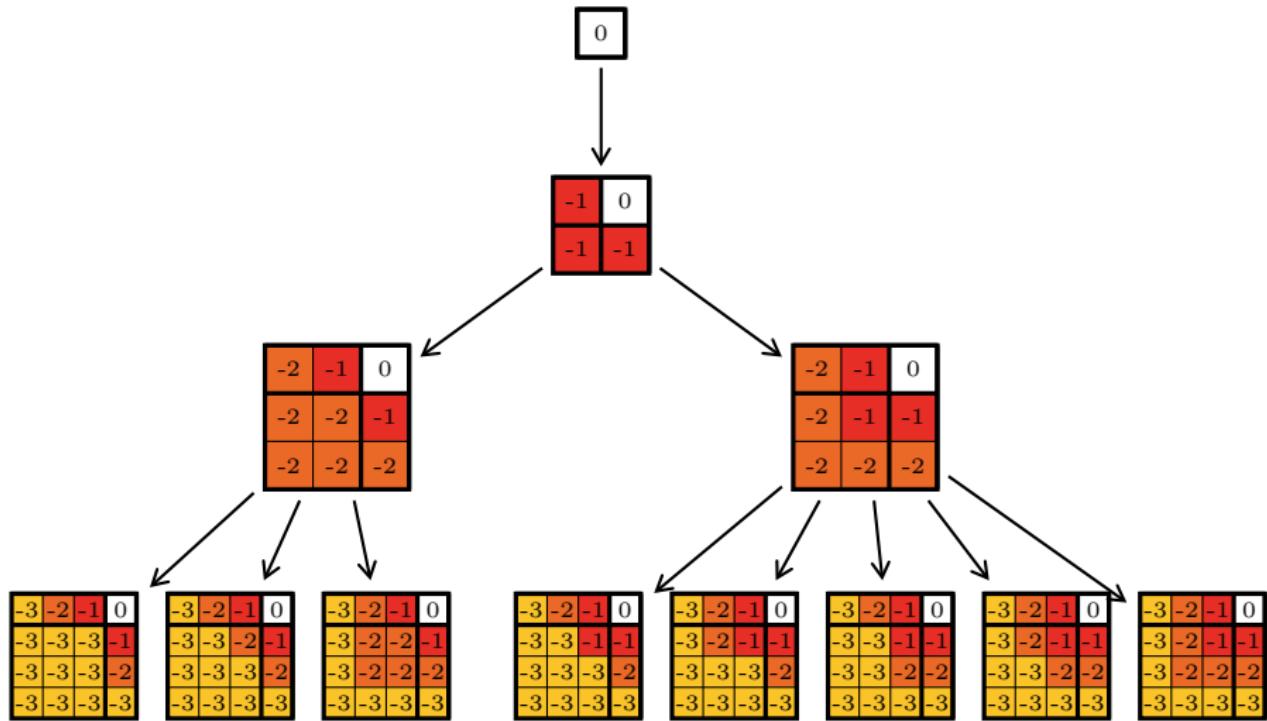
Tree of one-element Rees coextensions



Tree of one-element Rees coextensions



Tree of one-element Rees coextensions



Question

How to determine all the one-element Rees coextensions of a given finite, negative tomonoid?

Construction of one-element co-extensions

$$(S^2; \leqslant, \sim, (0,0)) \longrightarrow (\bar{S}^2; \leqslant, \dot{\sim}, (0,0))$$

tomonoid
partition

one-element
co-extensions

S ... starting tomonoid

S^* ... S with bottom element removed

\bar{S} = $S^* \dot{\cup} \{\perp, \alpha\}$, such that $\perp < \alpha < x$ for every $x \in S^*$
... *bottom-doubling extension* of S

Construction of one-element co-extensions

$(\bar{S}^2; \leqslant, \dot{\sim}, (0,0))$ is a one-element co-extension of $(S^2; \leqslant, \sim, (0,0))$ if:

- (E1) $\forall (a,b), (c,d) \in \mathcal{P}: (a,b) \sim (c,d) \Rightarrow (a,b) \dot{\sim} (c,d)$
- (E2) $\forall (a,b), (b,c) \in \mathcal{P}: (a,b) \sim d \text{ and } (b,c) \sim e \Rightarrow (d,c) \dot{\sim} (a,e)$
- (E3)
 - (a) $\forall (a,b) \in \mathcal{Q}: (-1,a) \sim d \Rightarrow (d,b) \dot{\sim} \perp$
 - (b) $\forall (a,b) \in \mathcal{Q}: (b,-1) \sim e \Rightarrow (a,e) \dot{\sim} \perp$
- (E4)
 - (a) $\forall b < 0: (\perp, 0) \dot{\sim} (0, \perp) \dot{\sim} (\alpha, b) \dot{\sim} (b, \alpha)$
 - (b) $(\alpha, 0) \dot{\sim} (0, \alpha)$
 - (c) $\forall (a,b), (c,d) \in \mathcal{Q}: (a,b) \leqslant (c,d) \dot{\sim} \perp \Rightarrow (a,b) \dot{\sim} \perp$

$$\begin{aligned}\mathcal{P} &= \{(a,b) \in S^2 \mid \text{there is } c \in S^\star \text{ such that } (a,b) \sim c\} \\ \mathcal{Q} &= \bar{S}^2 \setminus \mathcal{P}\end{aligned}$$

Example: Construction of a one-element co-extension

$$(S^2; \leqslant, \sim, (0,0)) \rightarrow (\bar{S}^2; \leqslant, \dot{\sim}, (0,0))$$

Step: Bottom-doubling extension

- S ... starting tomonoid
- S^* ... bottom element removed
- \bar{S} ... we add \perp and α s.t. $\perp < \alpha < x$ for every $x \in S^*$

Question

- Which pairs shall be assigned to \perp and which to α ?

S

-5	-4	-3	-2	-1	0
-5	-5	-5	-4	-3	-1
-5	-5	-5	-5	-5	-2
-5	-5	-5	-5	-5	-3
-5	-5	-5	-5	-5	-4
-5	-5	-5	-5	-5	-5

Example: Construction of a one-element co-extension

$$(S^2; \leqslant, \sim, (0,0)) \rightarrow (\bar{S}^2; \leqslant, \dot{\sim}, (0,0))$$

Step: Bottom-doubling extension

- S ... starting tomonoid
- S^* ... bottom element removed
- \bar{S} ... we add \perp and α s.t. $\perp < \alpha < x$ for every $x \in S^*$

Question

- Which pairs shall be assigned to \perp and which to α ?

	-4	-3	-2	-1	0
			-4	-3	-1
					-2
					-3
					-4

Example: Construction of a one-element co-extension

$$(S^2; \leqslant, \sim, (0,0)) \rightarrow (\bar{S}^2; \leqslant, \dot{\sim}, (0,0))$$

Step: Bottom-doubling extension

- S ... starting tomonoid
- S^* ... bottom element removed
- \bar{S} ... we add \perp and α s.t. $\perp < \alpha < x$ for every $x \in S^*$

Question

- Which pairs shall be assigned to \perp and which to α ?

		\bar{S}						
		S^*						
\perp	α	-4	-3	-2	-1	0		
				-4	-3	-1		
							-2	
								-3
							-4	
								α
								\perp

Example: Construction of a one-element co-extension

$$(S^2; \leqslant, \sim, (0,0)) \rightarrow (\bar{S}^2; \leqslant, \dot{\sim}, (0,0))$$

Step: Bottom-doubling extension

- S ... starting tomonoid
- S^* ... bottom element removed
- \bar{S} ... we add \perp and α s.t. $\perp < \alpha < x$ for every $x \in S^*$

Question

- Which pairs shall be assigned to \perp and which to α ?

\bar{S}
S*

\perp	α	-4	-3	-2	-1	0
			-4	-3	-1	
						-2
						-3
						-4
						α
						\perp

Example: Construction of a one-element co-extension

$$(S^2; \leqslant, \sim, (0,0)) \longrightarrow (\bar{S}^2; \leqslant, \dot{\sim}, (0,0))$$

Step (E1)

- $\forall (a, b), (c, d) \in \mathcal{P}$:

$$(a, b) \sim (c, d) \Rightarrow (a, b) \dot{\sim} (c, d)$$

\perp	α	-4	-3	-2	-1	0
			-4	-3	-1	
					-2	
						-3
					-4	
						α
						\perp

Example: Construction of a one-element co-extension

$$(S^2; \leqslant, \sim, (0,0)) \longrightarrow (\bar{S}^2; \leqslant, \dot{\sim}, (0,0))$$

Step (E1)

- $\forall (a, b), (c, d) \in \mathcal{P}$:

$$(a, b) \sim (c, d) \Rightarrow (a, b) \dot{\sim} (c, d)$$

\mathcal{P}

\perp	α	-4	-3	-2	1	0			
					-4	-3	-2	-1	
							-3	-2	
								-2	
									-1
									\perp

Example: Construction of a one-element co-extension

$$(S^2; \leqslant, \sim, (0,0)) \longrightarrow (\bar{S}^2; \leqslant, \dot{\sim}, (0,0))$$

Step (E2)

- $\forall (a, b), (b, c) \in \mathcal{P}$:

$$(a, b) \sim d \quad \text{and} \quad (b, c) \sim e$$

$$\Rightarrow (d, c) \dot{\sim} (a, e)$$

\perp	α	-4	-3	-2	-1	0
			-4	-3	-1	
					-2	
						-3
					-4	
						α
						\perp

Example: Construction of a one-element co-extension

$$(S^2; \leqslant, \sim, (0,0)) \longrightarrow (\bar{S}^2; \leqslant, \dot{\sim}, (0,0))$$

Step (E2)

- $\forall (a, b), (b, c) \in \mathcal{P}$:

$$(a, b) \sim d \quad \text{and} \quad (b, c) \sim e$$

$$\Rightarrow (d, c) \dot{\sim} (a, e)$$

							<i>b</i>
\perp	α	-4	-3	-2	-1	0	
				-4	-3	-1	
							-2
							-3
							-4
						α	
							\perp

a

Example: Construction of a one-element co-extension

$$(S^2; \trianglelefteq, \sim, (0,0)) \rightarrow (\bar{S}^2; \trianglelefteq, \dot{\sim}, (0,0))$$

Step (E2)

- $\forall (a, b), (b, c) \in \mathcal{P}:$

$$(a, b) \sim d \quad \text{and} \quad (b, c) \sim e$$

$$\Rightarrow (d, c) \dot{\sim} (a, e)$$

	α	-4	-3	-2	-1	0
				-4	-3	-1
						-2
						-3
						-4
						α
						1

Example: Construction of a one-element co-extension

$$(S^2; \trianglelefteq, \sim, (0,0)) \rightarrow (\bar{S}^2; \trianglelefteq, \dot{\sim}, (0,0))$$

Step (E2)

- $\forall (a, b), (b, c) \in \mathcal{P}:$

$$(a, b) \sim d \quad \text{and} \quad (b, c) \sim e$$

$$\Rightarrow (d, c) \dot{\sim} (a, e)$$

	\perp	α	-4	-3	-2	-1	0
					-4	-3	-1
							-2
							-3
							-4
							α
							\perp

Example: Construction of a one-element co-extension

$$(S^2; \trianglelefteq, \sim, (0,0)) \rightarrow (\bar{S}^2; \trianglelefteq, \dot{\sim}, (0,0))$$

Step (E2)

- $\forall (a, b), (b, c) \in \mathcal{P}:$

$$(a, b) \sim d \quad \text{and} \quad (b, c) \sim e$$

$$\Rightarrow (d, c) \dot{\sim} (a, e)$$

	\perp	α	-4	-3	-2	-1	0
					-4	-3	-1
							-2
							-3
							-4
							α
							\perp

Example: Construction of a one-element co-extension

$$(S^2; \leqslant, \sim, (0,0)) \longrightarrow (\bar{S}^2; \leqslant, \dot{\sim}, (0,0))$$

Step (E2)

- $\forall (a, b), (b, c) \in \mathcal{P}$:

$$(a, b) \sim d \quad \text{and} \quad (b, c) \sim e$$

$$\Rightarrow (d, c) \dot{\sim} (a, e)$$

		<i>c</i>	<i>b</i>	
\perp	α	-4	-3	-2
		-4	-3	-1
				-2
				-3
				-4
				α
				\perp

$a = b$

d

Example: Construction of a one-element co-extension

$$(S^2; \leqslant, \sim, (0,0)) \longrightarrow (\bar{S}^2; \leqslant, \dot{\sim}, (0,0))$$

Step (E2)

- $\forall (a, b), (b, c) \in \mathcal{P}$:

$$(a, b) \sim d \quad \text{and} \quad (b, c) \sim e$$

$$\Rightarrow (d, c) \dot{\sim} (a, e)$$

		<i>e</i>	<i>c</i>	<i>b</i>		
\perp	α	-4	-3	-2	-1	0
		-4	-3	-1		
					-2	
					-3	
					-4	
					α	
						\perp

$a = b$

d

Example: Construction of a one-element co-extension

$$(S^2; \leqslant, \sim, (0,0)) \longrightarrow (\bar{S}^2; \leqslant, \dot{\sim}, (0,0))$$

Step (E2)

- $\forall (a, b), (b, c) \in \mathcal{P}$:

$$(a, b) \sim d \quad \text{and} \quad (b, c) \sim e$$

$$\Rightarrow (d, c) \dot{\sim} (a, e)$$

		e	c	b		
\perp	α	-4	-3	-2	-1	0
			-4	-3	-1	
						-2
					-3	
					-4	
					α	
						\perp

$a = b$

d

Example: Construction of a one-element co-extension

$$(S^2; \leqslant, \sim, (0,0)) \longrightarrow (\bar{S}^2; \leqslant, \dot{\sim}, (0,0))$$

Step (E2)

- $\forall (a, b), (b, c) \in \mathcal{P}$:

$$(a, b) \sim d \quad \text{and} \quad (b, c) \sim e$$

$$\Rightarrow (d, c) \dot{\sim} (a, e)$$

\perp	α	-4	-3	-2	-1	0
				-4	-3	-1
						-2
						-3
						-4
						α
						\perp

A red wavy arrow starts at the cell containing -4 in the row labeled α and points to the cell containing -3 in the row labeled -1.

Example: Construction of a one-element co-extension

$$(S^2; \trianglelefteq, \sim, (0,0)) \rightarrow (\bar{S}^2; \trianglelefteq, \dot{\sim}, (0,0))$$

Step (E2)

- $\forall (a, b), (b, c) \in \mathcal{P}:$

$$\Rightarrow (d, c) \sim (a, e)$$

		\perp	α	-4	-3	-2	-1	0	b
									a
									-2
									-3
									-4
									α
									\perp

A diagram showing a grid of numbers from -4 to 0. The number -3 is highlighted with a green circle. A red circle is placed at the intersection of -4 and -3. A red wavy arrow points from this red circle to another red circle at the intersection of -3 and -2.

Example: Construction of a one-element co-extension

$$(S^2; \leqslant, \sim, (0,0)) \longrightarrow (\bar{S}^2; \leqslant, \dot{\sim}, (0,0))$$

Step (E2)

- $\forall (a, b), (b, c) \in \mathcal{P}$:

$$\begin{aligned} (a, b) \sim d &\quad \text{and} \quad (b, c) \sim e \\ \Rightarrow (d, c) \dot{\sim} (a, e) & \end{aligned} \quad = \color{blue}{c} \\ \color{green}{b} \end{math>$$

⊥	α	-4	-3	-2	-1	0

$a = \color{blue}{b}$

Example: Construction of a one-element co-extension

$$(S^2; \leqslant, \sim, (0,0)) \longrightarrow (\bar{S}^2; \leqslant, \dot{\sim}, (0,0))$$

Step (E2)

- $\forall (a, b), (b, c) \in \mathcal{P}$:

$$(a, b) \sim d \quad \text{and} \quad (b, c) \sim e \quad = \color{blue}{c}$$

$$\Rightarrow (d, c) \dot{\sim} (a, e)$$

\perp	α	-4	-3	-2	-1	0
				-4	-3	-1
					-2	
						-3
						-4
						α
						\perp

Example: Construction of a one-element co-extension

$$(S^2; \leqslant, \sim, (0,0)) \longrightarrow (\bar{S}^2; \leqslant, \dot{\sim}, (0,0))$$

Step (E2)

- $\forall (a, b), (b, c) \in \mathcal{P}$:

$$(a, b) \sim d \quad \text{and} \quad (b, c) \sim e$$

$$\Rightarrow (d, c) \dot{\sim} (a, e)$$

\perp	α	-4	-3	-2	-1	0
		○	○	-4	-3	-1
						-2
						-3
						-4
						α
						\perp

Example: Construction of a one-element co-extension

$$(S^2; \leqslant, \sim, (0,0)) \longrightarrow (\bar{S}^2; \leqslant, \dot{\sim}, (0,0))$$

Step (E3)

- $\forall (a, b) \in Q:$

- $(-1, a) \sim d \Rightarrow (d, b) \dot{\sim} \perp$
- $(b, -1) \sim e \Rightarrow (a, e) \dot{\sim} \perp$

\perp	α	-4	-3	-2	-1	0
		○	○	-4	-3	-1
						-2
						-3
						-4
						α
						\perp

Example: Construction of a one-element co-extension

$$(S^2; \leqslant, \sim, (0,0)) \longrightarrow (\bar{S}^2; \leqslant, \dot{\sim}, (0,0))$$

Step (E3)

- $\forall (a, b) \in Q:$

- $(-1, a) \sim d \Rightarrow (d, b) \dot{\sim} \perp$
- $(b, -1) \sim e \Rightarrow (a, e) \dot{\sim} \perp$

The diagram shows a grid of numbers arranged in a 7x8 pattern. The columns are labeled at the top as $\perp, \alpha, -4, -3, -2, -1, 0$. The rows are labeled on the right as $-1, -2, -3, -4, \alpha, \perp$. The grid contains several colored cells: blue, yellow, and red. A large blue shaded rectangular region covers the first four columns and the last four rows. Two specific cells are circled: one in the second row and third column (yellow), and another in the fourth row and fifth column (red). An arrow points from the letter Q to the bottom-left corner of the blue shaded area.

\perp	α	-4	-3	-2	-1	0	

Example: Construction of a one-element co-extension

$$(S^2; \leqslant, \sim, (0,0)) \longrightarrow (\bar{S}^2; \leqslant, \dot{\sim}, (0,0))$$

Step (E3)

- ▶ $\forall (a, b) \in Q:$

- ▶ $(-1, a) \sim d \Rightarrow (d, b) \dot{\sim} \perp$
- ▶ $(b, -1) \sim e \Rightarrow (a, e) \dot{\sim} \perp$

\perp	α	-4	-3	-2	-1	0
		○	■	-4	-3	-1
					■	-2
					○	-3
						-4
						α
						\perp

Example: Construction of a one-element co-extension

$$(S^2; \leqslant, \sim, (0,0)) \longrightarrow (\bar{S}^2; \leqslant, \dot{\sim}, (0,0))$$

Step (E3)

- $\forall (a, b) \in \mathcal{Q}$:

- $(-1, a) \sim d \Rightarrow (d, b) \dot{\sim} \perp$
- $(b, -1) \sim e \Rightarrow (a, e) \dot{\sim} \perp$

		b						
\perp	α	-4	-3	-2	-1	0		
		○	●					

Example: Construction of a one-element co-extension

$$(S^2; \trianglelefteq, \sim, (0,0)) \rightarrow (\bar{S}^2; \trianglelefteq, \dot{\sim}, (0,0))$$

Step (E3)

- $\forall (a, b) \in Q:$

- ▶ $(-1, a) \sim d \Rightarrow (d, b) \dot{\sim} \perp$
 - ▶ $(b, -1) \sim e \Rightarrow (a, e) \dot{\sim} \perp$

\perp	α	-4	-3	-2	-1	0
				-4	-3	-1
						-2
						-3
						-4
						α
						\perp

Example: Construction of a one-element co-extension

$$(S^2; \trianglelefteq, \sim, (0,0)) \rightarrow (\bar{S}^2; \trianglelefteq, \dot{\sim}, (0,0))$$

Step (E3)

- $\forall (a, b) \in Q:$

- $(-1, a) \sim d \Rightarrow (d, b) \dot{\sim} \perp$
 - $(b, -1) \sim e \Rightarrow (a, e) \dot{\sim} \perp$

\perp	α	-4	-3	-2	-1	0
				-4		-1
						-2
						-3
						-4
						α
						\perp

Example: Construction of a one-element co-extension

$$(S^2; \trianglelefteq, \sim, (0,0)) \rightarrow (\bar{S}^2; \trianglelefteq, \dot{\sim}, (0,0))$$

Step (E3)

- $\forall (a, b) \in Q:$

- $(-1, a) \sim d \Rightarrow (d, b) \dot{\sim} \perp$
 - $(b, -1) \sim e \Rightarrow (a, e) \dot{\sim} \perp$

\perp	α	-4	-3	-2	-1	0
				-4		-1
						-2
						-3
						-4
						α
						\perp

Example: Construction of a one-element co-extension

$$(S^2; \leqslant, \sim, (0,0)) \longrightarrow (\bar{S}^2; \leqslant, \dot{\sim}, (0,0))$$

Step (E3)

- $\forall (a, b) \in \mathcal{Q}$:

- $(-1, a) \sim d \Rightarrow (d, b) \dot{\sim} \perp$
- $(b, -1) \sim e \Rightarrow (a, e) \dot{\sim} \perp$

\perp	α	-4	-3	-2	-1	0	b	a	
		○	●		-4	-3		-1	

↓

\perp	○	○	○	○	-3				d

Example: Construction of a one-element co-extension

$$(S^2; \leqslant, \sim, (0,0)) \longrightarrow (\bar{S}^2; \leqslant, \dot{\sim}, (0,0))$$

Step (E3)

- $\forall (a, b) \in \mathcal{Q}$:

- $(-1, a) \sim d \Rightarrow (d, b) \dot{\sim} \perp$
- $(b, -1) \sim e \Rightarrow (a, e) \dot{\sim} \perp$

\perp	α	-4	-3	-2	-1	0	b
		○	●		-4	-3	-1
							-2
							-3
							-4
							α
							\perp

Example: Construction of a one-element co-extension

$$(S^2; \leqslant, \sim, (0,0)) \longrightarrow (\bar{S}^2; \leqslant, \dot{\sim}, (0,0))$$

Step (E3)

- $\forall (a, b) \in \mathcal{Q}$:

- $(-1, a) \sim d \Rightarrow (d, b) \dot{\sim} \perp$
- $(b, -1) \sim e \Rightarrow (a, e) \dot{\sim} \perp$

		-4	-3	-2	-1	0	
	\perp	α	-4	-3	-2	-1	0
	a						
	b						
	\perp						
	\perp						
	\perp						
	\perp						
	\perp						

The table shows a partial order \leqslant on the set S^2 . The elements are labeled with their coordinates: $\perp, \alpha, -4, -3, -2, -1, 0$. The order relations are indicated by colored circles: red circles for $\perp \leqslant a$ and yellow circles for $b \leqslant \perp$. Specifically, $\perp \leqslant \alpha$ (red), $\alpha \leqslant -4$ (yellow), $-4 \leqslant -3$ (yellow), $-3 \leqslant -2$ (yellow), $-2 \leqslant -1$ (yellow), $-1 \leqslant 0$ (yellow), and $\perp \leqslant \perp$ (red).

Example: Construction of a one-element co-extension

$$(S^2; \leqslant, \sim, (0,0)) \longrightarrow (\bar{S}^2; \leqslant, \dot{\sim}, (0,0))$$

Step (E3)

- $\forall (a, b) \in \mathcal{Q}$:

- $(-1, a) \sim d \Rightarrow (d, b) \dot{\sim} \perp$
- $(b, -1) \sim e \Rightarrow (a, e) \dot{\sim} \perp$

\perp	α	-4	-3	-2	-1	0	b
		○	●		-4	-3	-1
							-2
\perp							
							-3
							-4
							α
							\perp

a

b

Example: Construction of a one-element co-extension

$$(S^2; \trianglelefteq, \sim, (0,0)) \rightarrow (\bar{S}^2; \trianglelefteq, \dot{\sim}, (0,0))$$

Step (E3)

- $\forall (a, b) \in \mathcal{Q}:$

- ▶ $(-1, a) \sim d \Rightarrow (d, b) \dot{\sim} \perp$
 - ▶ $(b, -1) \sim e \Rightarrow (a, e) \dot{\sim} \perp$

\perp	α	-4	-3	-2	-1	0
				-4	-3	-1
						-2
					-3	
					-4	

Example: Construction of a one-element co-extension

$$(S^2; \leqslant, \sim, (0,0)) \longrightarrow (\bar{S}^2; \leqslant, \dot{\sim}, (0,0))$$

Step (E3)

- $\forall (a, b) \in \mathcal{Q}$:

- $(-1, a) \sim d \Rightarrow (d, b) \dot{\sim} \perp$
- $(b, -1) \sim e \Rightarrow (a, e) \dot{\sim} \perp$

		<i>a</i>	<i>b</i>			
\perp	α	-4	-3	-2	-1	0
		○	○	-4	-3	-1
				•	-2	
					-3	
					-4	
					α	
						\perp

Example: Construction of a one-element co-extension

$$(S^2; \leqslant, \sim, (0,0)) \longrightarrow (\bar{S}^2; \leqslant, \dot{\sim}, (0,0))$$

Step (E3)

- $\forall (a, b) \in \mathcal{Q}$:

- $(-1, a) \sim d \Rightarrow (d, b) \dot{\sim} \perp$
- $(b, -1) \sim e \Rightarrow (a, e) \dot{\sim} \perp$

		<i>a</i>	<i>b</i>			
\perp	α	-4	-3	-2	-1	0
		○	○	○	○	
					•	-2
						-3
						-4
						α
						\perp

Example: Construction of a one-element co-extension

$$(S^2; \leqslant, \sim, (0,0)) \longrightarrow (\bar{S}^2; \leqslant, \dot{\sim}, (0,0))$$

Step (E3)

- $\forall (a, b) \in \mathcal{Q}$:

- $(-1, a) \sim d \Rightarrow (d, b) \dot{\sim} \perp$
- $(b, -1) \sim e \Rightarrow (a, e) \dot{\sim} \perp$

		<i>a</i>	<i>b</i>			
\perp	α	-4	-3	-2	-1	0
		○	○	○	○	
					•	-2
						-3
\perp		○	○			-4
						<i>d</i>
						α
						\perp

Example: Construction of a one-element co-extension

$$(S^2; \leqslant, \sim, (0,0)) \longrightarrow (\bar{S}^2; \leqslant, \dot{\sim}, (0,0))$$

Step (E3)

- $\forall (a, b) \in \mathcal{Q}$:

- $(-1, a) \sim d \Rightarrow (d, b) \dot{\sim} \perp$
- $(b, -1) \sim e \Rightarrow (a, e) \dot{\sim} \perp$

		<i>a</i>	<i>b</i>			
\perp	α	-4	-3	-2	-1	0
		○	○	○	○	
					•	-2
						-3
					↓	
						-4
						α
						\perp

The table illustrates the construction of a one-element co-extension. The rows and columns are labeled with elements from the set S^2 . The first two rows represent the original structure $(S^2; \leqslant, \sim, (0,0))$, while the subsequent rows represent the extended structure $(\bar{S}^2; \leqslant, \dot{\sim}, (0,0))$. The elements are colored: \perp (light blue), α (light green), and the other elements (-4, -3, -2, -1, 0) are represented by circles. The diagonal from top-left to bottom-right consists of red circles. The row and column for element a are highlighted in blue, and the row and column for element d are highlighted in green. A yellow circle is at position (-3, -2). A blue dot is at position (-1, -2). A blue arrow points down from position (-2, -1) to (-1, -1). A red arrow points right from position (-1, -1) to (-1, -2).

Example: Construction of a one-element co-extension

$$(S^2; \leqslant, \sim, (0,0)) \longrightarrow (\bar{S}^2; \leqslant, \dot{\sim}, (0,0))$$

Step (E3)

- $\forall (a, b) \in \mathcal{Q}$:

- $(-1, a) \sim d \Rightarrow (d, b) \dot{\sim} \perp$
- $(b, -1) \sim e \Rightarrow (a, e) \dot{\sim} \perp$

\perp	α	-4	-3	-2	-1	0	b
		○	○	-4	-3	-1	
					•	-2	
						-3	
						-4	
						α	
						\perp	

a

Example: Construction of a one-element co-extension

$$(S^2; \leqslant, \sim, (0,0)) \longrightarrow (\bar{S}^2; \leqslant, \dot{\sim}, (0,0))$$

Step (E3)

- $\forall (a, b) \in \mathcal{Q}$:

- $(-1, a) \sim d \Rightarrow (d, b) \dot{\sim} \perp$
- $(b, -1) \sim e \Rightarrow (a, e) \dot{\sim} \perp$

\perp	α	-4	-3	-2	-1	0	b
							b
a							a
\perp							\perp
							-4
							α
							\perp

Example: Construction of a one-element co-extension

$$(S^2; \leqslant, \sim, (0,0)) \longrightarrow (\bar{S}^2; \leqslant, \dot{\sim}, (0,0))$$

Step (E3)

- $\forall (a, b) \in \mathcal{Q}$:

- $(-1, a) \sim d \Rightarrow (d, b) \dot{\sim} \perp$
- $(b, -1) \sim e \Rightarrow (a, e) \dot{\sim} \perp$

\perp	α	-4	-3	-2	-1	0	b
							b
a							a
\perp							\perp
							-3
							-4
							α
							\perp

Example: Construction of a one-element co-extension

$$(S^2; \leqslant, \sim, (0,0)) \longrightarrow (\bar{S}^2; \leqslant, \dot{\sim}, (0,0))$$

Step (E3)

- $\forall (a, b) \in Q:$

- $(-1, a) \sim d \Rightarrow (d, b) \dot{\sim} \perp$
- $(b, -1) \sim e \Rightarrow (a, e) \dot{\sim} \perp$

\perp	α	-4	-3	-2	-1	0	e	b

b
 a

Example: Construction of a one-element co-extension

$$(S^2; \leqslant, \sim, (0,0)) \longrightarrow (\bar{S}^2; \leqslant, \dot{\sim}, (0,0))$$

Step (E3)

- $\forall (a, b) \in \mathcal{Q}$:

- $(-1, a) \sim d \Rightarrow (d, b) \dot{\sim} \perp$
- $(b, -1) \sim e \Rightarrow (a, e) \dot{\sim} \perp$

\perp	α	-4	-3	-2	-1	0	e	b
			-3				-3	-1
			-4				-4	

a

Example: Construction of a one-element co-extension

$$(S^2; \leqslant, \sim, (0,0)) \longrightarrow (\bar{S}^2; \leqslant, \dot{\sim}, (0,0))$$

Step (E3)

- $\forall (a, b) \in \mathcal{Q}$:

- $(-1, a) \sim d \Rightarrow (d, b) \dot{\sim} \perp$
- $(b, -1) \sim e \Rightarrow (a, e) \dot{\sim} \perp$

\perp	α	-4	-3	-2	-1	0
		○	○	-4	-3	-1
				\perp		-2
				\perp	○	-3
					\perp	-4
						α
						\perp

Example: Construction of a one-element co-extension

$$(S^2; \leqslant, \sim, (0,0)) \longrightarrow (\bar{S}^2; \leqslant, \dot{\sim}, (0,0))$$

Step (E4)

(a) $\forall b < 0$:

$$(\perp, 0) \dot{\sim} (0, \perp) \dot{\sim} (\alpha, b) \dot{\sim} (b, \alpha)$$

(b) $(\alpha, 0) \dot{\sim} (0, \alpha)$

(c) $\forall (a, b), (c, d) \in \mathcal{Q}$:

$$(a, b) \leqslant (c, d) \dot{\sim} \perp \Rightarrow (a, b) \dot{\sim} \perp$$

\perp	α	-4	-3	-2	-1	0
		○	○	-4	-3	-1
				\perp		-2
				\perp	○	○
				\perp		-3
					\perp	-4
						α
						\perp

Example: Construction of a one-element co-extension

$$(S^2; \leqslant, \sim, (0,0)) \longrightarrow (\bar{S}^2; \leqslant, \dot{\sim}, (0,0))$$

Step (E4)

(a) $\forall b < 0$:

$$(\perp, 0) \dot{\sim} (0, \perp) \dot{\sim} (\alpha, b) \dot{\sim} (b, \alpha)$$

(b) $(\alpha, 0) \dot{\sim} (0, \alpha)$

(c) $\forall (a, b), (c, d) \in \mathcal{Q}$:

$$(a, b) \leqslant (c, d) \dot{\sim} \perp \Rightarrow (a, b) \dot{\sim} \perp$$

\perp	α	-4	-3	-2	-1	0
\perp	\perp	○	○	-4	-3	-1
\perp	\perp		\perp			-2
\perp	\perp		\perp	○	○	-3
\perp	\perp			\perp		-4
\perp	\perp	\perp	\perp	\perp	\perp	α
\perp	\perp	\perp	\perp	\perp	\perp	\perp

Example: Construction of a one-element co-extension

$$(S^2; \leqslant, \sim, (0,0)) \longrightarrow (\bar{S}^2; \leqslant, \dot{\sim}, (0,0))$$

Step (E4)

(a) $\forall b < 0$:

$$(\perp, 0) \dot{\sim} (0, \perp) \dot{\sim} (\alpha, b) \dot{\sim} (b, \alpha)$$

(b) $(\alpha, 0) \dot{\sim} (0, \alpha)$

(c) $\forall (a, b), (c, d) \in \mathcal{Q}$:

$$(a, b) \leqslant (c, d) \dot{\sim} \perp \Rightarrow (a, b) \dot{\sim} \perp$$

\perp	α	-4	-3	-2	-1	0
\perp	\perp	○	○	-4	-3	-1
\perp	\perp		\perp			-2
\perp	\perp		\perp	○	○	-3
\perp	\perp			\perp		-4
\perp	\perp	\perp	\perp	\perp	\perp	α
\perp	\perp	\perp	\perp	\perp	\perp	\perp

Example: Construction of a one-element co-extension

$$(S^2; \leqslant, \sim, (0,0)) \longrightarrow (\bar{S}^2; \leqslant, \dot{\sim}, (0,0))$$

Step (E4)

(a) $\forall b < 0$:

$$(\perp, 0) \dot{\sim} (0, \perp) \dot{\sim} (\alpha, b) \dot{\sim} (b, \alpha)$$

(b) $(\alpha, 0) \dot{\sim} (0, \alpha)$

(c) $\forall (a, b), (c, d) \in \mathcal{Q}$:

$$(a, b) \leqslant (c, d) \dot{\sim} \perp \Rightarrow (a, b) \dot{\sim} \perp$$

\perp	α	-4	-3	-2	-1	0
\perp	\perp	○	○	-4	-3	-1
\perp	\perp	\perp	\perp			-2
\perp	\perp	\perp	\perp			
\perp	\perp	\perp	\perp	○	○	-3
\perp	\perp	\perp	\perp	\perp	\perp	-4
\perp	\perp	\perp	\perp	\perp	\perp	α
\perp	\perp	\perp	\perp	\perp	\perp	\perp

Example: Construction of a one-element co-extension

$$(S^2; \leqslant, \sim, (0,0)) \longrightarrow (\bar{S}^2; \leqslant, \dot{\sim}, (0,0))$$

(E1) $\forall (a,b), (c,d) \in \mathcal{P}$:

$$(a,b) \sim (c,d) \Rightarrow (a,b) \dot{\sim} (c,d)$$

(E2) $\forall (a,b), (b,c) \in \mathcal{P}$:

$$\begin{aligned} (a,b) \sim d \quad \text{and} \quad (b,c) \sim e \\ \Rightarrow (d,c) \dot{\sim} (a,e) \end{aligned}$$

(E3) $\forall (a,b) \in \mathcal{Q}$:

$$\begin{aligned} \blacktriangleright (-1,a) \sim d \Rightarrow (d,b) \dot{\sim} \perp \\ \blacktriangleright (b,-1) \sim e \Rightarrow (a,e) \dot{\sim} \perp \end{aligned}$$

(E4) (a) $\forall b < 0$:

$$(\perp, 0) \dot{\sim} (0, \perp) \dot{\sim} (\alpha, b) \dot{\sim} (b, \alpha)$$

(b) $(\alpha, 0) \dot{\sim} (0, \alpha)$

(c) $\forall (a,b), (c,d) \in \mathcal{Q}$:

$$(a,b) \leqslant (c,d) \dot{\sim} \perp \Rightarrow (a,b) \dot{\sim} \perp$$

\perp	α	-4	-3	-2	-1	0
\perp	\perp	○	○	-4	-3	-1
\perp	\perp	\perp	-	-	-	-2
\perp	\perp	\perp	\perp	○	○	-3
\perp	\perp	\perp	\perp	\perp	\perp	-4
\perp	\perp	\perp	\perp	\perp	\perp	α
\perp	\perp	\perp	\perp	\perp	\perp	\perp

Example: Construction of a one-element co-extension

\perp	α	-4	-3	-2	-1	0
\perp	\perp	\perp	\perp	-4	-3	-1
\perp	\perp	\perp	\perp	\perp	\perp	-2
\perp	\perp	\perp	\perp	\perp	\perp	-3
\perp	\perp	\perp	\perp	\perp	\perp	-4
\perp	\perp	\perp	\perp	\perp	\perp	α
\perp	\perp	\perp	\perp	\perp	\perp	\perp

\perp	α	-4	-3	-2	-1	0
\perp	\perp	\perp	\perp	-4	-3	-1
\perp	\perp	\perp	\perp	\perp	α	-2
\perp	\perp	\perp	\perp	\perp	\perp	-3
\perp	\perp	\perp	\perp	\perp	\perp	-4
\perp	\perp	\perp	\perp	\perp	\perp	α
\perp	\perp	\perp	\perp	\perp	\perp	\perp

\perp	α	-4	-3	-2	-1	0
\perp	\perp	\perp	\perp	-4	-3	-1
\perp	\perp	\perp	\perp	\perp	α	-2
\perp	\perp	\perp	\perp	\perp	\perp	-3
\perp	\perp	\perp	\perp	\perp	\perp	-4
\perp	\perp	\perp	\perp	\perp	\perp	α
\perp	\perp	\perp	\perp	\perp	\perp	\perp

\perp	α	-4	-3	-2	-1	0
\perp	\perp	\perp	α	-4	-3	-1
\perp	\perp	\perp	\perp	\perp	α	-2
\perp	\perp	\perp	\perp	\perp	\perp	-3
\perp	\perp	\perp	\perp	\perp	\perp	-4
\perp	\perp	\perp	\perp	\perp	\perp	α
\perp	\perp	\perp	\perp	\perp	\perp	\perp

\perp	α	-4	-3	-2	-1	0
\perp	\perp	\perp	α	-4	-3	-1
\perp	\perp	\perp	\perp	α	α	-2
\perp	\perp	\perp	\perp	\perp	α	-3
\perp	\perp	\perp	\perp	\perp	\perp	-4
\perp	\perp	\perp	\perp	\perp	\perp	α
\perp	\perp	\perp	\perp	\perp	\perp	\perp

\perp	α	-4	-3	-2	-1	0
\perp	α	α	α	-4	-3	-1
\perp	\perp	\perp	\perp	α	α	-2
\perp	\perp	\perp	\perp	\perp	α	-3
\perp	\perp	\perp	\perp	\perp	\perp	-4
\perp	\perp	\perp	\perp	\perp	\perp	α
\perp	\perp	\perp	\perp	\perp	\perp	\perp

Concluding slide

aa

Thank you for your kind attention!