Cross-connections and variants of $\mathscr{T}_X$

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2. My sincere thanks to A. R. Rajan, University of Kerala, India, and M. V. Volkov, Ural Federal University, Russia, for their support and guidance.

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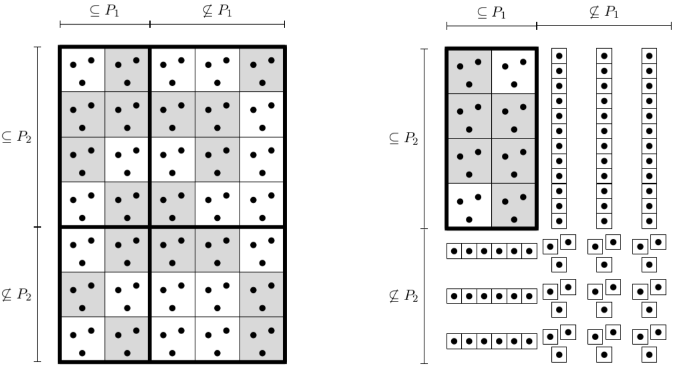
- In 2015, Dolinka and East explored the structure of $\mathscr{T}_X^\theta$, its idempotent generated subsemigroup, its regular part, its ideals etc.
- The following subsets of $\mathscr{T}_X$ was crucial in their discussion,

$$P_1 = \{a \in \mathscr{T}_X : a\theta \mathscr{R} \theta\} \qquad P_2 = \{a \in \mathscr{T}_X : \theta a \mathscr{L} \theta\}.$$

- They gave the following diagram to show how a typical $\mathscr{D}$-class of $\mathscr{T}_X$ (in the left), breaks up to the corresponding $\mathscr{D}$-classes of $\mathscr{T}_X^\theta$.



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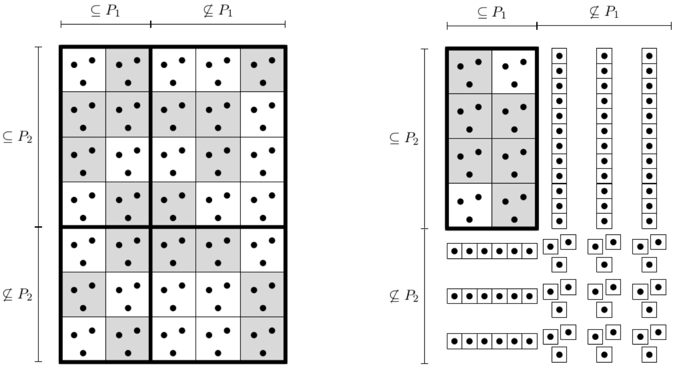

- In this talk, we discuss the ideal structure of $Reg(\mathscr{T}_X^\theta)$—the regular part of $\mathscr{T}_X^\theta$ (i.e., $P_1 \cap P_2$).

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- In 1994, Nambooripad (1994) extended the latter approach to arbitrary regular semigroups using cross-connected categories.

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- The purpose is two fold.
- First, this semigroup provides a concrete setting where all the abstract notions of cross-connection theory has transparent, yet non-trivial meanings.

- Second, we give an alternate path to the structural description of $Reg(\mathscr{T}_X^\theta)$ given by Dolinka and East, using subsets and partitions of X.

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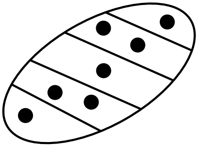
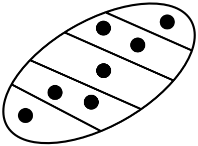
- Let A be a subset of X and α an equivalence relation (or a partition) on X.



- We say the subset A saturates the partition α if each α-class contains at least one element of A.



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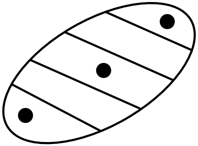

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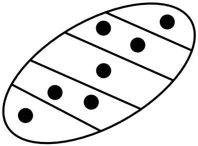

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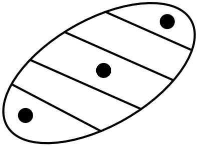

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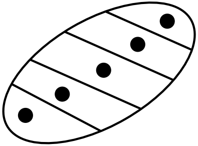

- So, the subset A is a cross-section of the partition α, if A saturates α and α separates A.

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- So, in the case of $Reg(\mathscr{T}_X^\theta)$, the second category is determined by the partitions of X saturated by Im θ, say Π_θ.
- The cross-connection construction also involves certain intermediary regular semigroups arising from these categories.

- In $Reg(\mathscr{T}_X^\theta)$, they are isomorphic to the sets P_1 and P_2 discussed earlier, seen as subsemigroups of $\mathscr{T}_X$.

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- Thus, we can realise $Reg(\mathscr{T}_X^{\theta})$ as a cross-connection semigroup

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- This representation gives the following description of the biordered set and sandwich sets of $Reg(\mathscr{T}_X^\theta)$.

- In $Reg(\mathscr{T}_X^\theta)$, the idempotents are given by :

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- Thus, we can describe the biordered set, completely in terms of subsets and partitions.
- Then the Sandwich set $\mathcal{S}(A, \pi) = \mathcal{S}((A, \pi'), (A', \pi))$ is given by

$$\mathcal{S}(A, \pi) = \{(X, \sigma) : X \text{ is a cross-section of } \pi \text{ and } A \text{ is a cross-section of } \sigma\}$$

where $A, A', X \in \mathscr{P}_\theta$ and $\pi, \pi', \sigma \in \Pi_\theta$.

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- So, this dual approach may help in extending the results to a more general class of semigroups.
- How we may extend this approach to the entire variant semigroup is another question.
- We believe that, a solution to this problem may shed some light into the much more general problem of the cross-connection construction of arbitrary semigroups.