Cross-connections and variants of $T_X$

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My sincere thanks to A. R. Rajan, University of Kerala, India, and M. V. Volkov, Ural Federal University, Russia, for their support and guidance.
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$$\alpha * \beta = \alpha \cdot \theta \cdot \beta$$

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In 2015, Dolinka and East explored the structure of $T^\theta_X$, its idempotent generated subsemigroup, its regular part, its ideals etc.

The following subsets of $T_X$ was crucial in their discussion,

$$P_1 = \{ a \in T_X : a\theta R \theta \} \quad P_2 = \{ a \in T_X : \theta a L \theta \}.$$
They gave the following diagram to show how a typical $\mathcal{D}$-class of $\mathcal{T}_X$ (in the left), breaks up to the corresponding $\mathcal{D}$-classes of $\mathcal{T}^\theta_X$.
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In this talk, we discuss the ideal structure of $\text{Reg}(\mathcal{I}^\theta_X)$—the regular part of $\mathcal{I}^\theta_X$ (i.e., $P_1 \cap P_2$).
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In 1994, Nambooripad (1994) extended the latter approach to arbitrary regular semigroups using cross-connected categories.
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The purpose is two fold.

First, this semigroup provides a concrete setting where all the abstract notions of cross-connection theory has transparent, yet non-trivial meanings.
Second, we give an alternate path to the structural description of $\text{Reg}(\mathcal{T}_X^\theta)$ given by Dolinka and East, using subsets and partitions of $X$. This, in turn, suggests that their results obtained in the specific case of this variant semigroup, is much more universal in nature. This discussion also yields a description of the biorder structure of $\text{Reg}(\mathcal{T}_X^\theta)$. It can be seen that the principal ideals (or equivalently Green's relations) in $\mathcal{T}_X$ and its variants are determined by the subsets and partitions of $X$. So, naturally, the description of the ideal structure of these semigroups involves subsets and partitions. For that, we borrow the terminology of Dolinka and East. Let $A$ be a subset of $X$ and $\alpha$ an equivalence relation (or a partition) on $X$. 
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Let \( A \) be a subset of \( X \) and \( \alpha \) an equivalence relation (or a partition) on \( X \).
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- So, the subset $A$ is a cross-section of the partition $\alpha$, if $A$ saturates $\alpha$ and $\alpha$ separates $A$. 
Now, this leads us to define a category $\mathcal{P}_\theta$ with the set of objects as:

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So, in the case of $\text{Reg}(\mathcal{I}_X^\theta)$, the second category is determined by the partitions of $X$ saturated by $\text{Im} \theta$, say $\Pi_\theta$. 
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The cross-connection construction also involves certain intermediary regular semigroups arising from these categories.
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In $Reg(\mathcal{T}_X^\theta)$, this functor, say $\Gamma_\theta$, is completely characterised by the sandwich element, $\theta$.

Thus, we can realise $Reg(\mathcal{T}_X^\theta)$ as a cross-connection semigroup

$$(\Pi_\theta, \mathcal{P}_\theta; \Gamma_\theta) = \{(\theta a, a\theta) : a \in Reg(\mathcal{T}_X^\theta)\}.$$
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This representation gives the following description of the biordered set and sandwich sets of $\text{Reg}(\mathcal{T}_X^\theta)$. 


In $Reg(\mathcal{T}_X)$, the idempotents are given by:

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Then the Sandwich set $S(A, \pi) = S((A, \pi'), (A', \pi))$ is given by

$$S(A, \pi) = \{(X, \sigma) : X \text{ is a cross-section of } \pi \text{ and } A \text{ is a cross-section of } \sigma \}$$

where $A, A', X \in \mathcal{P}_\theta$ and $\pi, \pi', \sigma \in \Pi_\theta$. 
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We believe that, a solution to this problem may shed some light into the much more general problem of the cross-connection construction of arbitrary semigroups.