Restriction semigroups and $\lambda$-Zappa-Szép products

Rida-e Zenab

University of York

AAA94+NSAC, 15-18 June 2017

Based on joint work with Victoria Gould
Definitions: Restriction Semigroups, Zappa-Szép Products and Categories
Definitions: Restriction Semigroups, Zappa-Szép Products and Categories

\(\lambda\)-Zappa-Szép products of inverse semigroups
Definitions: Restriction Semigroups, Zappa-Szép Products and Categories

- \(\lambda\)-Zappa-Szép products of inverse semigroups
- \(\lambda\)-Zappa-Szép products of restriction semigroups
Definitions: Restriction Semigroups, Zappa-Szép Products and Categories

- $\lambda$-Zappa-Szép products of inverse semigroups
- $\lambda$-Zappa-Szép products of restriction semigroups
- $\lambda$-Semidirect products of restriction semigroups
Restriction semigroups

Left restriction semigroups form a variety of unary semigroups, that is, semigroups equipped with an additional unary operation, denoted by $\cdot$. The identities that define a left restriction semigroup $S$ are:

\[ a + a = a; \]
\[ a + b + a = b + a + a; \]
\[ (a + b) + a = a + b + a; \]
\[ ab + a = (ab) + a; \]

We put $E = f_1a_2Sg_3$; then $E$ is a semilattice known as the semilattice of projections of $S$.

Dually right restriction semigroups form a variety of unary semigroups. In this case the unary operation is denoted by $\cdot$. A restriction semigroup is a bi-unary semigroup $S$ which is both left restriction and right restriction and which also satisfies the linking identities

\[ (a + b) + a = a + b; \]
\[ (a) + a = a; \]
Left restriction semigroups form a variety of unary semigroups, that is, semigroups equipped with an additional unary operation, denoted by $^+$. The identities that define a left restriction semigroup $S$ are:

$$a^+ a = a, a^+ b^+ = b^+ a^+, (a^+ b)^+ = a^+ b^+, ab^+ = (ab)^+ a.$$ 

We put

$$E = \{ a^+ : a \in S \},$$

then $E$ is a semilattice known as the semilattice of projections of $S$. 


Left restriction semigroups form a variety of unary semigroups, that is, semigroups equipped with an additional unary operation, denoted by $^+$. The identities that define a left restriction semigroup $S$ are:

$$a^+ a = a, a^+ b^+ = b^+ a^+, (a^+ b)^+ = a^+ b^+, ab^+ = (ab)^+ a.$$ 

We put

$$E = \{a^+ : a \in S\},$$

then $E$ is a semilattice known as the semilattice of projections of $S$. Dually right restriction semigroups form a variety of unary semigroups. In this case the unary operation is denoted by $^*$. 
Left restriction semigroups form a variety of unary semigroups, that is, semigroups equipped with an additional unary operation, denoted by $^+$. The identities that define a left restriction semigroup $S$ are:

$$a^+ a = a, a^+ b^+ = b^+ a^+, (a^+ b)^+ = a^+ b^+, ab^+ = (ab)^+ a.$$ 

We put

$$E = \{a^+ : a \in S\},$$

then $E$ is a semilattice known as the semilattice of projections of $S$.

Dually right restriction semigroups form a variety of unary semigroups. In this case the unary operation is denoted by $^*$.

A restriction semigroup is a bi-unary semigroup $S$ which is both left restriction and right restriction and which also satisfies the linking identities

$$(a^+)^* = a^+ \text{ and } (a^*)^+ = a^*.$$
Let $S$ and $T$ be semigroups and suppose that we have maps

$$T \times S \to S, \quad (t, s) \mapsto t \cdot s$$

$$T \times S \to T, \quad (t, s) \mapsto t^s$$

such that for all $s, s' \in S, t, t' \in T$, the following hold:
Let $S$ and $T$ be semigroups and suppose that we have maps

$$T \times S \to S, \quad (t, s) \mapsto t \cdot s$$
$$T \times S \to T, \quad (t, s) \mapsto t^s$$

such that for all $s, s' \in S, t, t' \in T$, the following hold:

(ZS1) $tt' \cdot s = t \cdot (t' \cdot s)$;

(ZS2) $t \cdot (ss') = (t \cdot s)(t^s \cdot s')$;

(ZS3) $(t^s)^{s'} = t^{ss'}$;

(ZS4) $(tt')^s = t^{t' \cdot s} t'^s$. 

Define a binary operation on $S \times T$ by $(s; t) \cdot (s'; t') = (s \cdot (t' \cdot s); t \cdot s' \cdot t')$.

Then $S \triangleleft \triangleleft T$ is a semigroup, known as the Zappa-Szép product of $S$ and $T$ and denoted by $S \triangleleft \triangleleft T$. 

Zappa-Szép products of semigroups
Let $S$ and $T$ be semigroups and suppose that we have maps

$$T \times S \rightarrow S, \quad (t, s) \mapsto t \cdot s$$
$$T \times S \rightarrow T, \quad (t, s) \mapsto t^s$$

such that for all $s, s' \in S$, $t, t' \in T$, the following hold:

1. **(ZS1)** $tt' \cdot s = t \cdot (t' \cdot s)$;
2. **(ZS2)** $t \cdot (ss') = (t \cdot s)(t^s \cdot s')$;
3. **(ZS3)** $(t^s)^{s'} = t^{ss'}$;
4. **(ZS4)** $(tt')^s = t^{t' \cdot s} t'$. 

Define a binary operation on $S \times T$ by

$$(s, t)(s', t') = (s(t \cdot s'), t^s t').$$

Then $S \times T$ is a semigroup, known as the Zappa-Szép product of $S$ and $T$ and denoted by $S \bowtie T$. 
If $S$ and $T$ are monoids then we insist that the following four axioms also hold:

1. $1_S 1_S = 1_S$
2. $1_T s = s$
3. $t 1_S = t$
4. $1_T 1_T = 1_T$
If $S$ and $T$ are monoids then we insist that the following four axioms also hold:

$(ZS5)$ \( t \cdot 1_S = 1_S \); \hspace{1em} (ZS7) \( 1_T \cdot s = s \);

$(ZS6)$ \( t^{1_S} = t \); \hspace{1em} (ZS8) \( 1^s_T = 1_T \).

Then $S \bowtie T$ is a monoid with identity $(1_S, 1_T)$. 
By a category $\mathbf{C} = (C, \cdot, d, r)$, we mean a small category in the standard sense, where $\cdot$ is a partial binary operation on $C$ and $d, r : C \to C$.
By a category $\mathbf{C} = (\mathbf{C}, \cdot, d, r)$, we mean a small category in the standard sense, where $\cdot$ is a partial binary operation on $\mathbf{C}$ and $d, r : \mathbf{C} \to \mathbf{C}$. 
By a category $C = (C, \cdot, d, r)$, we mean a small category in the standard sense, where $\cdot$ is a partial binary operation on $C$ and $d, r : C \to C$. 

\[
\begin{align*}
d(x) & \quad r(x) = d(y) \\
x & \quad y \\
xy &
\end{align*}
\]
By a category $\mathbf{C} = (C, \cdot, d, r)$, we mean a *small category* in the standard sense, where $\cdot$ is a partial binary operation on $C$ and $d, r : C \to C$. We say that $\mathbf{C}$ is a **groupoid**

$$
\begin{align*}
d(x) &= xx^{-1} & r(x) &= x^{-1}x
\end{align*}
$$
An ordered category \((C, \cdot, d, r, \leq)\) is a category \((C, \cdot, d, r)\) with a partial order on \(C\) such that \(\leq\) is compatible with multiplication and if \(x \leq y\), then

\[
d(x) \leq d(y) \text{ and } r(x) \leq r(y),
\]

and possessing restrictions and co-restrictions.
An ordered category \((C, \cdot, d, r, \leq)\) is a category \((C, \cdot, d, r)\) with a partial order on \(C\) such that \(\leq\) is compatible with multiplication and if \(x \leq y\), then

\[d(x) \leq d(y) \text{ and } r(x) \leq r(y),\]

and possessing restrictions and co-restrictions.

In an ordered groupoid, \(\leq\) must be compatible with \(x \mapsto x^{-1}\).
An ordered category \((C, \cdot, d, r, \leq)\) is a category \((C, \cdot, d, r)\) with a partial order on \(C\) such that \(\leq\) is compatible with multiplication and if \(x \leq y\), then

\[
d(x) \leq d(y) \text{ and } r(x) \leq r(y),
\]

and possessing restrictions and co-restrictions.

In an ordered groupoid, \(\leq\) must be compatible with \(x \mapsto x^{-1}\).

An inductive category (groupoid) is an ordered category (groupoid) in which identities form a semilattice.
The Ehresmann-Schein-Nambooripad Theorem

The category of inverse semigroups and homomorphisms is isomorphic to the category of inductive groupoids and inductive functors.

Theorem (M. V. Lawson)
The category of restriction semigroups and (2,1,1)-morphisms is isomorphic to the category of inductive categories and inductive functors.
The Ehresmann-Schein-Nambooripad Theorem

The ESN Theorem

The category of inverse semigroups and homomorphisms is isomorphic to the category of inductive groupoids and inductive functors.
The Ehresmann-Schein-Nambooripad Theorem

The ESN Theorem
The category of inverse semigroups and homomorphisms is isomorphic to the category of inductive groupoids and inductive functors.

Theorem (M. V. Lawson)
The category of restriction semigroups and (2,1,1)-morphisms is isomorphic to the category of inductive categories and inductive functors.
Theorem (Gibert and Wazzan)

Let $Z \triangleleft \triangleleft T$ be a Zappa-Szép product of inverse semigroups $S$ and $T$. Then $B \triangleleft \triangleleft (Z)$ is a groupoid under the restriction of the binary operation in $Z$.

If the action of $S$ on $T$ is trivial, then $B \circ (S \circ T) = f(a; t)$.
Theorem (Gibert and Wazzan)
Let $Z = S \bowtie T$ be a Zappa-Szép product of inverse semigroups $S$ and $T$. Then
Theorem (Gibert and Wazzan)
Let \( Z = S \bowtie T \) be a Zappa-Szép product of inverse semigroups \( S \) and \( T \). Then

\[
B_\bowtie(Z) = \{(a, t) \in S \times T : \quad tt^{-1} \cdot a^{-1} = a^{-1}, \quad tt^{-1} \cdot a^{-1}a = a^{-1}a, \\
(t^{-1})a^{-1}a = t^{-1}, \quad (tt^{-1})a^{-1}a = tt^{-1}\}
\]

is a groupoid under the restriction of the binary operation in \( Z \).
Theorem (Gibert and Wazzan)
Let $Z = S \bowtie T$ be a Zappa-Szép product of inverse semigroups $S$ and $T$. Then

$$B_{\bowtie}(Z) = \{(a, t) \in S \times T : \quad tt^{-1} \cdot a^{-1} = a^{-1}, \quad tt^{-1} \cdot a^{-1}a = a^{-1}a, \quad (t^{-1})a^{-1}a = t^{-1}, \quad (tt^{-1})a^{-1}a = tt^{-1}\}$$

is a groupoid under the restriction of the binary operation in $Z$.

If the action of $S$ on $T$ is trivial, then

$$B_{\times}(S \times T) = \{(a, t) \in S \times T : \quad tt^{-1} \cdot a^{-1} = a^{-1}, \quad tt^{-1} \cdot a^{-1}a = a^{-1}a\}$$

is the Billhardt’s $\lambda$-semidirect of two inverse semigroups $S$ and $T$. 

\[\text{\lambda-Zappa-Szép product of inverse semigroups}\]
Theorem (Gilbert and Wazzan)

Let $E$ be a semilattice, $G$ be a group and $Z = E \triangleleft \triangleleft G$. Suppose that $(ZS7)$ holds. Then $B \triangleleft \triangleleft (Z) = f(e; g) \in E \times G : (g_1 e) = g_1 g$ is an inductive groupoid under the restriction of the binary operation in $Z$ with partial order defined by the rule $(e; g) \leq (f; h), e = f$ and $g = h$. 
Theorem (Gilbert and Wazzan)

Let $E$ be a semilattice, $G$ be a group and $Z = E \bowtie G$. Suppose that $(ZS7)\ 1 \cdot e = e$ holds. Then
Theorem (Gilbert and Wazzan)
Let $E$ be a semilattice, $G$ be a group and $Z = E \bowtie G$. Suppose that (ZS7) $1 \cdot e = e$ holds. Then

$$B_{\bowtie}(Z) = \{(e, g) \in E \times G : (g^{-1})^e = g^{-1}\}$$

is an inductive groupoid under the restriction of the binary operation in $Z$ with partial order defined by the rule

$$(e, g) \leq (f, h) \iff e \leq f \text{ and } g = h^{h^{-1} \cdot e}.$$
Notion of double action

Let $S$ and $T$ be restriction semigroups and suppose that $Z = S \bowtie T$. 

{(1)}

{(2)}

(CP1)

(CP2)
Notion of double action

Let $S$ and $T$ be restriction semigroups and suppose that $Z = S \bowtie T$. We say that $S$ and $T$ act \textit{doubly} on each other if we have two extra maps

$$S \times T \to T, (s, t) \mapsto s^t \text{ and } S \times T \to S, (s, t) \mapsto s \circ t$$

such that for all $s, s' \in S, t, t' \in T$:

$$\begin{align*}
(1) \quad ss' t &= s(\cdot s')t; \\
(2) \quad s \circ tt' &= (s \circ t) \circ t'
\end{align*}$$
Notion of double action

Let $S$ and $T$ be restriction semigroups and suppose that $Z = S \bowtie T$. We say that $S$ and $T$ act *doubly* on each other if we have two extra maps

$$S \times T \to T, (s, t) \mapsto ^s t$$ and $$S \times T \to S, (s, t) \mapsto s \circ t$$

such that for all $s, s' \in S, t, t' \in T$:

(1) $^{ss'} t = ^s (s' t)$;
(2) $s \circ tt' = (s \circ t) \circ t'$

and actions satisfies the following compatibility conditions

$$(^s t)^s = t^{s^*} = s^* t$$  
$$^s (t^s) = t^{s^+} = s^+ t.$$  

(CP1)

and

$$(t \cdot s) \circ t = s \circ t^* = t^* \cdot s$$  
$$t \cdot (s \circ t) = s \circ t^+ = t^+ \cdot s.$$  

(CP2)
Let $S$ and $T$ be restriction semigroups and $Z = S \bowtie T$. Suppose that $S$ and $T$ are acting doubly on each other satisfying (CP1) and (CP2).
Let $S$ and $T$ be restriction semigroups and $Z = S \bowtie T$. Suppose that $S$ and $T$ are acting doubly on each other satisfying (CP1) and (CP2). Let

$$V = \{(a, t) \in S \times T : t^+ \cdot a^* = a^*, (t^+)^a^* = t^+, a t^+ \cdot a = a, t^{a^* \circ t} = t\}.$$  

We aim to show that $V$ is category.
Some observations

In inverse case:

\[ t^b b^{-1} = t \Rightarrow \begin{cases} (t \cdot b)^{-1}(t \cdot b) = t^b \cdot b^{-1} b \\ (t \cdot b)(t \cdot b)^{-1} = t \cdot bb^{-1} \end{cases} \]

and

\[ t^{-1} t \cdot b = b \Rightarrow \begin{cases} (t^b)^{-1}t^b = (t^{-1}t)^b \\ t^b(t^b)^{-1} = (tt^{-1})^{t \cdot b}. \end{cases} \]
Some observations

In inverse case:

\[ t^{bb^{-1}} = t \Rightarrow \begin{cases} (t \cdot b)^{-1}(t \cdot b) = t^{b} \cdot b^{-1} b \\ (t \cdot b)(t \cdot b)^{-1} = t \cdot bb^{-1} \end{cases} \]

and

\[ t^{-1} t \cdot b = b \Rightarrow \begin{cases} (t^{b})^{-1}t^{b} = (t^{-1}t)^{b} \\ t^{b}(t^{b})^{-1} = (tt^{-1})^{t \cdot b}. \]

Reformulated to restriction case

\[ t^{b^{+}} = t \Rightarrow \begin{cases} (t \cdot b)^{*} = t^{b} \cdot b^{*} \\ (t \cdot b)^{+} = t \cdot b^{+} \end{cases} \] (A)

and

\[ t^{*} \cdot b = b \Rightarrow \begin{cases} (t^{b})^{*} = (t^{*})^{b} \\ (t^{b})^{+} = (t^{+})^{t \cdot b}. \] (B)
Construction of a category
Construction of a category

Theorem (V. Gould and RZ)
Let $S$ and $T$ be restriction semigroups and suppose that $Z = S \bowtie T$ is Zappa-Szép product of $S$ and $T$. Suppose that the actions satisfies (CP1) and (CP2).
Construction of a category

**Theorem (V. Gould and RZ)**
Let $S$ and $T$ be restriction semigroups and suppose that $Z = S \bowtie T$ is Zappa-Szép product of $S$ and $T$. Suppose that the actions satisfies $(CP1)$ and $(CP2)$. Let

$$V = \{(a, t) \in S \times T : t^+ \cdot a^* = a^*, \quad (t^+)a^* = t^+, \quad a t^+ \cdot a = a, \quad t a^* \circ t = t\}.$$ 

For $(a, t) \in V$, we suppose that, $a^* \circ t \in E_S$ and $a t^+ \in E_T$. Also suppose that $(A)$ and $(B)$ hold. Then $V$ is a category under the restriction of the binary operation in $Z$. 

We consider the Zappa-Szép product of a semilattice and a monoid. We suppose that in this case (ZS7) $1 \cdot e = e$ and (ZS8) $1^e = 1$ holds. As a monoid $T$ is reduced restriction semigroup with

$$t^+ = 1 = t^* \text{ for all } t \in T,$$
We consider the Zappa-Szép product of a semilattice and a monoid. We suppose that in this case (ZS7) \( 1 \cdot e = e \) and (ZS8) \( 1^e = 1 \) holds. As a monoid \( T \) is reduced restriction semigroup with
\[
t^+ = 1 = t^* \quad \text{for all } t \in T,
\]
therefore
\[
V = \{(a, t) \in S \times T : t^+ \cdot a^* = a^*, (t^+)a^* = t^+, a t^+ \cdot a = a, t a^* \circ t = t\}
\]
reduces to
\[
V' = \{(e, t) \in E \times T : t = t^{e \circ t}\}
\]
We consider the Zappa-Szép product of a semilattice and a monoid. We suppose that in this case (ZS7) $1 \cdot e = e$ and (ZS8) $1^e = 1$ holds. As a monoid $T$ is reduced restriction semigroup with $t^+ = 1 = t^*$ for all $t \in T$, therefore

$$V = \{(a, t) \in S \times T : t^+ \cdot a^* = a^*, (t^+)^a = t^+, a^t^+ \cdot a = a, t^{a^* \circ t} = t\}$$

reduces to

$$V' = \{(e, t) \in E \times T : t = t^{e \circ t}\}$$

and (CP1) and (CP2) read as

$$e = (t \cdot e) \circ t = t \cdot (e \circ t), \quad \text{(CP3)}$$

$$t^e = t^e = t^e (t^e). \quad \text{(CP4)}$$
Also note that in this special case (A) and (B) are satisfied trivially.
Also note that in this special case (A) and (B) are satisfied trivially.

**Theorem (V. Gould and RZ)**
Let \( Z = E \bowtie T \) be a Zappa-Szép product of a semilattice \( E \) and a monoid \( T \). Suppose that \( 1 \cdot e = e, \ 1^e = 1 \) and the action of \( T \) on \( E \) and \( E \) on \( T \) satisfies (CP3) and (CP4), respectively. Also suppose that

\[
e \leq f \Rightarrow t^f \cdot e = t \cdot e.
\]
Also note that in this special case (A) and (B) are satisfied trivially.

**Theorem (V. Gould and RZ)**

Let $Z = E \bowtie T$ be a Zappa-Szép product of a semilattice $E$ and a monoid $T$. Suppose that $1 \cdot e = e$, $1^e = 1$ and the action of $T$ on $E$ and $E$ on $T$ satisfies (CP3) and (CP4), respectively. Also suppose that

$$e \leq f \Rightarrow t^f \cdot e = t \cdot e.$$ 

Then

$$V' = \{(e, t) \in E \times T : t = t^{e\circ t}\}$$

is an inductive category under the restriction of binary operation in $Z$ with partial order $\leq$ defined by

$$(e, s) \leq (f, t)$$

if and only if $e \leq f$ and $s = t^{e\circ t}$. 
Obtaining a restriction semigroup

We define a pseudo product on $V'$ by the rule

$$(e, s) \otimes (f, t) = ((e, s)|_{r(e,s) \wedge d(f,t)}) (r(e,s) \wedge d(f,t)|_{(f, t)})$$

where

$$r(e, s) \wedge d(f, t) = (e \circ s, 1) \wedge (f, 1) = ((e \circ s)f, 1)$$

so that $V'$ is a restriction semigroup with multiplication defined by

$$(e, s)(f, t) = (e(s \cdot f), s^f t^{(e \circ s)t}).$$
**Theorem (V. Gould and RZ)** Let $S$ and $T$ be restriction semigroups and suppose that $T$ acts doubly on $S$ satisfying (CP2).
Theorem (V. Gould and RZ) Let $S$ and $T$ be restriction semigroups and suppose that $T$ acts doubly on $S$ satisfying (CP2). Let $Z = S \rtimes T$ and put

$$P = S \rtimes^\lambda T = \{(a, t) \in S \times T : t^+ \cdot a = a\}.$$
Theorem (V. Gould and RZ) Let $S$ and $T$ be restriction semigroups and suppose that $T$ acts doubly on $S$ satisfying (CP2). Let $Z = S \rtimes T$ and put

$$P = S \rtimes^\lambda T = \{(a, t) \in S \times T : t^+ \cdot a = a\}.$$ 

Then $P$ is an inductive category with partial order $\leq$ defined by

$$(a, t) \leq (b, u) \quad \text{if and only if} \quad a \leq t^+ \cdot b, \ t \leq u.$$
When one of the action is trivial ($\lambda$-semidirect products)

**Theorem (V. Gould and RZ)** Let $S$ and $T$ be restriction semigroups and suppose that $T$ acts doubly on $S$ satisfying (CP2). Let $Z = S \rtimes T$ and put

$$P = S \rtimes^\lambda T = \{(a, t) \in S \times T : t^+ \cdot a = a\}.$$  

Then $P$ is an inductive category with partial order $\leq$ defined by

$$(a, t) \leq (b, u) \text{ if and only if } a \leq t^+ \cdot b, \ t \leq u.$$  

By defining a pseudo product on our inductive category we obtain a restriction semigroup $P = S \rtimes^\lambda T$ with multiplication defined by

$$(a, t)(b, u) = \left(\left((tu)^+ \cdot a\right)(t \cdot b), tu\right).$$
Thank You!