Restriction semigroups and λ-Zappa-Szép products

Rida-e Zenab

University of York

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Based on joint work with Victoria Gould

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Restriction semigroups

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Left restriction semigroups form a variety of unary semigroups, that is, semigroups equipped with an additional unary operation, denoted by $^+$. The identities that define a left restriction semigroup S are:

$$a^+a = a, a^+b^+ = b^+a^+, (a^+b)^+ = a^+b^+, ab^+ = (ab)^+a.$$

We put

$$E = \{a^+ : a \in S\},$$

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Dually right restriction semigroups form a variety of unary semigroups. In this case the unary operation is denoted by *.

A restriction semigroup is a bi-unary semigroup S which is both left restriction and right restriction and which also satisfies the linking identities

$$(a^+)^* = a^+ \text{ and } (a^*)^+ = a^*.$$

Zappa-Szép products of semigroups

Let S and T be semigroups and suppose that we have maps

$$\begin{array}{ll} T \times S \rightarrow S, & (t, s) \mapsto t \cdot s \\ T \times S \rightarrow T, & (t, s) \mapsto t^s \end{array}$$

such that for all $s, s' \in S, t, t' \in T$, the following hold:

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such that for all $s, s' \in S, t, t' \in T$, the following hold:

(ZS1) $tt' \cdot s = t \cdot (t' \cdot s)$;
(ZS2) $t \cdot (ss') = (t \cdot s)(t^s \cdot s')$;
(ZS3) $(t^s)^{s'} = t^{ss'}$;
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(ZS4) $(tt')^s = t^{t' \cdot s} t'^s$.

Define a binary operation on $S \times T$ by

$$(s, t)(s', t') = (s(t \cdot s'), t^{s'} t').$$

Then $S \times T$ is a semigroup, known as the Zappa-Szép product of S and T and denoted by $S \bowtie T$.

Zappa-Szép products of monoids

If S and T are monoids then we insist that the following four axioms also hold:

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$$\text{(ZS5) } t \cdot 1_S = 1_S; \quad \text{(ZS7) } 1_T \cdot s = s;$$
$$\text{(ZS6) } t^{1_S} = t; \quad \text{(ZS8) } 1_T^s = 1_T.$$

Then $S \bowtie T$ is a monoid with identity $(1_S, 1_T)$.

Categories

By a category $\mathbf{C} = (C, \cdot, \mathbf{d}, \mathbf{r})$, we mean a *small category* in the standard sense, where $\cdot$ is a partial binary operation on C and $\mathbf{d}, \mathbf{r} : C \to C$.

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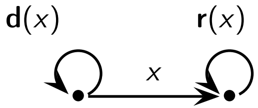

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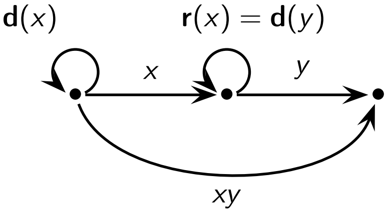
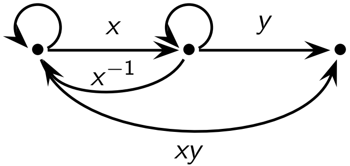


By a category $\mathbf{C} = (C, \cdot, \mathbf{d}, \mathbf{r})$, we mean a *small category* in the standard sense, where $\cdot$ is a partial binary operation on C and $\mathbf{d}, \mathbf{r} : C \to C$. We say that $\mathbf{C}$ is a groupoid

$$\mathbf{d}(x) = xx^{-1} \quad \mathbf{r}(x) = x^{-1}x$$



Inductive categories

- An ordered category $(C, \cdot, \mathbf{d}, \mathbf{r}, \leq)$ is a category $(C, \cdot, \mathbf{d}, \mathbf{r})$ with a partial order on C such that $\leq$ is compatible with multiplication and if $x \leq y$, then

  $$\mathbf{d}(x) \leq \mathbf{d}(y) \text{ and } \mathbf{r}(x) \leq \mathbf{r}(y),$$

  and possessing restrictions and co-restrictions.

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- In an ordered groupoid, $\leq$ must be compatible with $x \mapsto x^{-1}$.
- An inductive category (groupoid) is an ordered category (groupoid) in which identities form a semilattice.

The Ehresmann-Schein-Nambooripad Theorem

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Theorem (M. V. Lawson)
The category of restriction semigroups and (2,1,1)-morphisms is isomorphic to the category of inductive categories and inductive functors.

λ-Zappa-Szép product of inverse semigroups

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Theorem (Gibert and Wazzan)

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$$B_{\bowtie}(Z) = \{(a,t) \in S \times T : \; tt^{-1} \cdot a^{-1} = a^{-1},\, tt^{-1} \cdot a^{-1}a = a^{-1}a, \\ (t^{-1})^{a^{-1}a} = t^{-1}, (tt^{-1})^{a^{-1}a} = tt^{-1}\}$$

is a groupoid under the restriction of the binary operation in Z.

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is a groupoid under the restriction of the binary operation in Z.

If the action of S on T is trivial, then

$$B_{\rtimes}(S \rtimes T) = \{(a,t) \in S \times T : tt^{-1} \cdot a^{-1} = a^{-1},\, tt^{-1} \cdot a^{-1}a = a^{-1}a\}$$

is the Billhardt's λ-semidirect of two inverse semigroups S and T.

λ-Zappa-Szép product of inverse semigroups: Special case

Theorem (Gilbert and Wazzan)

Let E be a semilattice, G be a group and $Z = E \bowtie G$. Suppose that (ZS7) $1 \cdot e = e$ holds. Then

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$$B_{\bowtie}(Z) = \{(e, g) \in E \times G : (g^{-1})^e = g^{-1}\}$$

is an inductive groupoid under the restriction of the binary operation in Z with partial order defined by the rule

$$(e, g) \leq (f, h) \Leftrightarrow e \leq f \text{ and } g = h^{h^{-1} \cdot e}.$$

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$$S \times T \to T,\ (s,t) \mapsto {}^{s}t \text{ and } S \times T \to S,\ (s,t) \mapsto s \circ t$$

such that for all $s, s' \in S, t, t' \in T$:

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$$(1)\ {}^{ss'}t = {}^{s}({}^{s'}t); \quad (2)\ s \circ tt' = (s \circ t) \circ t'$$

and actions satisfies the following compatibility conditions

$$\begin{aligned} ({}^{s}t)^{s} &= t^{s^*} = {}^{s^*}t \\ {}^{s}(t^{s}) &= t^{s^+} = {}^{s^+}t. \end{aligned} \qquad \text{(CP1)}$$

and

$$\begin{aligned} (t \cdot s) \circ t &= s \circ t^* = t^* \cdot s \\ t \cdot (s \circ t) &= s \circ t^+ = t^+ \cdot s \end{aligned} \qquad \text{(CP2)}$$

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$$V = \{(a, t) \in S \times T : t^{+} \cdot a^{*} = a^{*}, (t^{+})^{a^{*}} = t^{+}, {}^{a}t^{+} \cdot a = a, t^{a^{*} \circ t} = t\}.$$

We aim to show that V is category.

Some observations

In inverse case:

$$t^{bb^{-1}} = t \Rightarrow \begin{cases} (t \cdot b)^{-1}(t \cdot b) = t^b \cdot b^{-1}b \\ (t \cdot b)(t \cdot b)^{-1} = t \cdot bb^{-1} \end{cases}$$

and

$$t^{-1}t \cdot b = b \Rightarrow \begin{cases} (t^b)^{-1}t^b = (t^{-1}t)^b \\ t^b(t^b)^{-1} = (tt^{-1})^{t \cdot b}. \end{cases}$$

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Reformulated to restriction case

$$t^{b^+} = t \Rightarrow \begin{cases} (t \cdot b)^* = t^b \cdot b^* \\ (t \cdot b)^+ = t \cdot b^+ \end{cases} \quad \text{(A)}$$

and

$$t^* \cdot b = b \Rightarrow \begin{cases} (t^b)^* = (t^*)^b \\ (t^b)^+ = (t^+)^{t \cdot b}. \end{cases} \quad \text{(B)}$$

Construction of a category

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Theorem (V. Gould and RZ)
Let S and T be restriction semigroups and suppose that $Z = S \bowtie T$ is Zappa-Szép product of S and T. Suppose that the actions satisfies (CP1) and (CP2).

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$$V = \{(a,t) \in S \times T : \begin{array}{rcl} t^+ \cdot a^* & = & a^*, \\ {}^a t^+ \cdot a & = & a, \end{array} \quad \begin{array}{rcl} (t^+)^{a^*} & = & t^+, \\ t^{a^* \circ t} & = & t\}. \end{array}$$

For $(a,t) \in V$, we suppose that, $a^* \circ t \in E_S$ and ${}^a t^+ \in E_T$. Also suppose that (A) and (B) hold. Then V is a category under the restriction of the binary operation in Z.

λ-Zappa-Szép product of restriction semigroups: Special case

We consider the Zappa-Szép product of a semilattice and a monoid. We suppose that in this case (ZS7) $1 \cdot e = e$ and (ZS8)$1^e = 1$ holds. As a monoid T is reduced restriction semigroup with

$$t^+ = 1 = t^* \quad \text{for all } t \in T,$$

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reduces to

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and (CP1) and (CP2) read as

$$e = (t \cdot e) \circ t = t \cdot (e \circ t), \tag{CP3}$$

$$({}^{e}t)^{e} = t^{e} = {}^{e}t =^{e} (t^{e}). \tag{CP4}$$

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Theorem (V. Gould and RZ)

Let $Z = E \bowtie T$ be a Zappa-Szép product of a semilattice E and a monoid T. Suppose that $1 \cdot e = e$, $1^e = 1$ and the action of T on E and E on T satisfies (CP3) and (CP4), respectively. Also suppose that

$$e \leq f \Rightarrow t^f \cdot e = t \cdot e.$$

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Theorem (V. Gould and RZ)

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$$e \leq f \Rightarrow t^f \cdot e = t \cdot e.$$

Then

$$V' = \{(e, t) \in E \times T : t = t^{e \circ t}\}$$

is an inductive category under the restriction of binary operation in Z with partial order $\leq$ defined by

$$(e, s) \leq (f, t) \text{ if and only if } e \leq f \text{ and } s = t^{e \circ t}.$$

Obtaining a restriction semigroup

We define a pseudo product on V' by the rule

$$(e,s) \otimes (f,t) = \left((e,s)|_{\mathbf{r}(e,s)\wedge \mathbf{d}(f,t)}\right)\left({}_{\mathbf{r}(e,s)\wedge \mathbf{d}(f,t)}|(f,t)\right)$$

where

$$\mathbf{r}(e,s) \wedge \mathbf{d}(f,t) = (e \circ s, 1) \wedge (f,1) = ((e \circ s)f, 1)$$

so that V' is a restriction semigroup with multiplication defined by

$$(e,s)(f,t) = \left(e(s \cdot f), s^f t^{(e \circ st)}\right).$$

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Then P is an inductive category with partial order $\leq$ defined by

$$(a, t) \leq (b, u) \quad \text{if and only if} \quad a \leq t^{+} \cdot b\,, \; t \leq u.$$

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Then P is an inductive category with partial order $\leq$ defined by

$$(a,t) \leq (b,u) \quad \text{if and only if} \quad a \leq t^{+} \cdot b\,, \; t \leq u.$$

By defining a pseudo product on our inductive category we obtain a restriction semigroup $P = S \rtimes^{\lambda} T$ with multiplication defined by

$$(a,t)(b,u) = \Big(\big((tu)^{+} \cdot a\big)\big(t \cdot b\big), tu\Big).$$

Thank You!