Sandwich semigroups in locally small categories

I. Dolinka[1] I. Đurđev[1,2] J. East[3] P. Honyam[4]
K. Sangkhanan[4] J. Sanwong[4] W. Sommanaee[4]

[1]University of Novi Sad

[2]Mathematical Institute
of the Serbian Academy of Sciences and Arts

[3]Western Sydney University

[4]Chiang Mai Univeristy

AAA94 + NSAC 2017

Outline

I. Dolinka, I. Đurđev, J. East, P. Honyam, K. Sangkhanan, J. Sanwong, W. Sommanaee

The Main Idea

The Main Idea

Let

The Main Idea

Let

- $\mathcal{S}$ be a locally small category,

The Main Idea

Let

- $\mathcal{S}$ be a locally small category,
- $\mathcal{S}_{XY}$ be the set of $X \to Y$ morphisms for $X, Y \quad \mathcal{S}$,

The Main Idea

Let

- $\mathcal{S}$ be a locally small category,
- $\mathcal{S}_{XY}$ be the set of $X \to Y$ morphisms for $X, Y \quad \mathcal{S}$,
- $a \quad \mathcal{S}_{YX}$ be a fixed morphism for a chosen pair $X, Y \quad \mathcal{S}$.

I. Dolinka, I. Đurđev, J. East, P. Honyam, K. Sangkhanan, J. Sanwong, W. Sommanaee

The Main Idea

Let

- $\mathcal{S}$ be a locally small category,
- $\mathcal{S}_{XY}$ be the set of $X \to Y$ morphisms for $X, Y \quad \mathcal{S}$,
- $a \quad \mathcal{S}_{YX}$ be a fixed morphism for a chosen pair $X, Y \quad \mathcal{S}$.

Then

- $\mathcal{S}^a_{XY} = (\mathcal{S}_{XY}, \star_a)$ is a semigroup, where

$$f \star_a g = fag \text{ for } f, g \quad \mathcal{S}_{XY}.$$

The Main Idea

Let

- $\mathcal{S}$ be a locally small category,
- $\mathcal{S}_{XY}$ be the set of $X \to Y$ morphisms for $X, Y \quad \mathcal{S}$,
- $a \quad \mathcal{S}_{YX}$ be a fixed morphism for a chosen pair $X, Y \quad \mathcal{S}$.

Then

- $\mathcal{S}^a_{XY} = (\mathcal{S}_{XY}, \star_a)$ is a semigroup, where

$$f \star_a g = fag \text{ for } f, g \quad \mathcal{S}_{XY}.$$

Examples

T^a_{XY},

The Main Idea

Let

- $\mathcal{S}$ be a locally small category,
- $\mathcal{S}_{XY}$ be the set of $X \to Y$ morphisms for $X, Y \in \mathcal{S}$,
- $a \in \mathcal{S}_{YX}$ be a fixed morphism for a chosen pair $X, Y \in \mathcal{S}$.

Then

- $\mathcal{S}^a_{XY} = (\mathcal{S}_{XY}, \star_a)$ is a semigroup, where

$$f \star_a g = fag \text{ for } f, g \in \mathcal{S}_{XY}.$$

Examples

T^a_{XY}, PT^a_{XY},

The Main Idea

Let

- $\mathcal{S}$ be a locally small category,
- $\mathcal{S}_{XY}$ be the set of $X \to Y$ morphisms for $X, Y \quad \mathcal{S}$,
- $a \quad \mathcal{S}_{YX}$ be a fixed morphism for a chosen pair $X, Y \quad \mathcal{S}$.

Then

- $\mathcal{S}_{XY}^{a} = (\mathcal{S}_{XY}, \star_a)$ is a semigroup, where

$$f \star_a g = fag \text{ for } f, g \quad \mathcal{S}_{XY}.$$

Examples

T_{XY}^{a}, PT_{XY}^{a}, variants of semigroups,

I. Dolinka,
I. Đurđev,
J. East,
P. Honyam,
K. Sangkhanan,
J. Sanwong,
W. Sommanaee

The Main Idea

Let

- $\mathcal{S}$ be a locally small category,
- $\mathcal{S}_{XY}$ be the set of $X \to Y$ morphisms for $X, Y \quad \mathcal{S}$,
- $a \quad \mathcal{S}_{YX}$ be a fixed morphism for a chosen pair $X, Y \quad \mathcal{S}$.

Then

- $\mathcal{S}^a_{XY} = (\mathcal{S}_{XY}, \star_a)$ is a semigroup, where

$$f \star_a g = fag \text{ for } f, g \quad \mathcal{S}_{XY}.$$

Examples

T^a_{XY}, PT^a_{XY}, variants of semigroups, $M^A_{m,n}$ ...

Partial semigroup $(S, \cdot, I, \lambda, \rho)$

Partial semigroup $(S, \cdot, I, \lambda, \rho)$

S and I are classes,

Partial semigroup $(S, \cdot, I, \lambda, \rho)$

S and I are classes, $(x, y) \to x \cdot y$ is a partial binary operation

Partial semigroup $(S, \cdot, I, \lambda, \rho)$

S and I are classes, $(x, y) \to x \cdot y$ is a partial binary operation and $\lambda, \rho : S \to I$ are functions, such that, for all $x, y, z \quad S$,

Partial semigroup $(S, \cdot, I, \lambda, \rho)$

S and I are classes, $(x, y) \to x \cdot y$ is a partial binary operation and $\lambda, \rho : S \to I$ are functions, such that, for all $x, y, z \quad S$,

(i) $x \cdot y$ is defined if and only if $x\rho = y\lambda$,

Partial semigroup $(S, \cdot, I, \lambda, \rho)$

S and I are classes, $(x, y) \to x \cdot y$ is a partial binary operation and $\lambda, \rho : S \to I$ are functions, such that, for all $x, y, z \in S$,

(i) $x \cdot y$ is defined if and only if $x\rho = y\lambda$,

(ii) if $x \cdot y$ is defined, then $(x \cdot y)\lambda = x\lambda$ and $(x \cdot y)\rho = y\rho$,

Partial semigroup $(S, \cdot, I, \lambda, \rho)$

S and I are classes, $(x, y) \to x \cdot y$ is a partial binary operation and $\lambda, \rho : S \to I$ are functions, such that, for all $x, y, z \quad S$,

(i) $x \cdot y$ is defined if and only if $x\rho = y\lambda$,

(ii) if $x \cdot y$ is defined, then $(x \cdot y)\lambda = x\lambda$ and $(x \cdot y)\rho = y\rho$,

(iii) if $x \cdot y$ and $y \cdot z$ are defined, then $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,

Partial semigroup $(S, \cdot, I, \lambda, \rho)$

S and I are classes, $(x, y) \to x \cdot y$ is a partial binary operation and $\lambda, \rho : S \to I$ are functions, such that, for all $x, y, z \quad S$,

(i) $x \cdot y$ is defined if and only if $x\rho = y\lambda$,

(ii) if $x \cdot y$ is defined, then $(x \cdot y)\lambda = x\lambda$ and $(x \cdot y)\rho = y\rho$,

(iii) if $x \cdot y$ and $y \cdot z$ are defined, then $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,

(iv) for all $i, j \quad I$, $S_{ij} = \{x \quad S : x\lambda = i,\ x\rho = j\}$ is a set.

Partial semigroup $(S, \cdot, I, \lambda, \rho)$

S and I are classes, $(x, y) \to x \cdot y$ is a partial binary operation and $\lambda, \rho : S \to I$ are functions, such that, for all $x, y, z \quad S$,

(i) $x \cdot y$ is defined if and only if $x\rho = y\lambda$,

(ii) if $x \cdot y$ is defined, then $(x \cdot y)\lambda = x\lambda$ and $(x \cdot y)\rho = y\rho$,

(iii) if $x \cdot y$ and $y \cdot z$ are defined, then $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,

(iv) for all $i, j \quad I$, $S_{ij} = \{x \quad S : x\lambda = i,\ x\rho = j\}$ is a set.

a partial semigroup is *monoidal* if in addition to (i)–(iv),

Partial semigroup $(S, \cdot, I, \lambda, \rho)$

S and I are classes, $(x, y) \to x \cdot y$ is a partial binary operation and $\lambda, \rho : S \to I$ are functions, such that, for all $x, y, z \quad S$,

(i) $x \cdot y$ is defined if and only if $x\rho = y\lambda$,

(ii) if $x \cdot y$ is defined, then $(x \cdot y)\lambda = x\lambda$ and $(x \cdot y)\rho = y\rho$,

(iii) if $x \cdot y$ and $y \cdot z$ are defined, then $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,

(iv) for all $i, j \quad I$, $S_{ij} = \{x \quad S : x\lambda = i,\ x\rho = j\}$ is a set.

a partial semigroup is *monoidal* if in addition to (i)–(iv),

(v) there exists a function $I \to S : i \to e_i$ such that, for all $x \quad S$, $x \cdot e_{x\rho} = x = e_{x\lambda} \cdot x$.

Partial semigroup $(S, \cdot, I, \lambda, \rho)$

S and I are classes, $(x, y) \mapsto x \cdot y$ is a partial binary operation and $\lambda, \rho : S \to I$ are functions, such that, for all $x, y, z \in S$,

(i) $x \cdot y$ is defined if and only if $x\rho = y\lambda$,

(ii) if $x \cdot y$ is defined, then $(x \cdot y)\lambda = x\lambda$ and $(x \cdot y)\rho = y\rho$,

(iii) if $x \cdot y$ and $y \cdot z$ are defined, then $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,

(iv) for all $i, j \in I$, $S_{ij} = \{x \in S : x\lambda = i,\ x\rho = j\}$ is a set.

a partial semigroup is *monoidal* if in addition to (i)–(iv),

(v) there exists a function $I \to S : i \mapsto e_i$ such that, for all $x \in S$, $x \cdot e_{x\rho} = x = e_{x\lambda} \cdot x$.

Furthermore, $\mathrm{Reg}(S) = \{x \in S : x = xyx\ (\exists y \in S)\}$.

Partial transformations

Partial transformations

- For sets $A, B \in \mathbf{Set}$, we write

$$\mathbf{PT}_{AB} = \{f : f \text{ is a function } C \to B \text{ for some } C \subseteq A\}.$$

Partial transformations

- For sets $A, B \quad \mathbf{Set}$, we write

$$\mathbf{PT}_{AB} = \{f : f \text{ is a function } C \to B \text{ for some } C \quad A\}.$$

- Furthermore,

$$PT \; = \{(A, f, B) : A, B \quad \mathbf{Set},\ f \quad \mathbf{PT}_{AB}\} \text{ and}$$

Partial transformations

- For sets $A, B \quad \textbf{Set}$, we write

$$\textbf{PT}_{AB} = \{f : f \text{ is a function } C \to B \text{ for some } C \quad A\}.$$

- Furthermore,

$$PT \quad = \{(A, f, B) : A, B \quad \textbf{Set},\ f \quad \textbf{PT}_{AB}\} \text{ and}$$
$$PT_{AB} = \{(A, f, B) : f \quad \textbf{PT}_{AB}\},$$

Partial transformations

- For sets $A, B \quad \mathbf{Set}$, we write

$$\mathbf{PT}_{AB} = \{f : f \text{ is a function } C \to B \text{ for some } C \quad A\}.$$

- Furthermore,

$$PT \;=\{(A, f, B) : A, B \quad \mathbf{Set},\ f \quad \mathbf{PT}_{AB}\} \text{ and}$$
$$PT_{AB} =\{(A, f, B) : f \quad \mathbf{PT}_{AB}\},$$

with partial product on PT, $\cdot$, defined by

$$(A, f, B) \cdot (C, g, D) = \begin{cases} (A, fg, D) & \text{if } B = C \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Partial transformations

- For sets $A, B \quad \mathbf{Set}$, we write

$$\mathbf{PT}_{AB} = \{f : f \text{ is a function } C \to B \text{ for some } C \quad A\}.$$

- Furthermore,

$$PT \; = \{(A, f, B) : A, B \quad \mathbf{Set},\ f \quad \mathbf{PT}_{AB}\} \text{ and}$$
$$PT_{AB} = \{(A, f, B) : f \quad \mathbf{PT}_{AB}\},$$

with partial product on PT , $\cdot$, defined by

$$(A, f, B) \cdot (C, g, D) = \begin{cases} (A, fg, D) & \text{if } B = C \\ \text{undefined} & \text{otherwise.} \end{cases}$$

- Finally, we define mappings

$$\lambda : PT \; \to \mathbf{Set} : (A, f, B) \to A \text{ and}$$

I. Dolinka, I. Đurđev, J. East, P. Honyam, K. Sangkhanan, J. Sanwong, W. Sommanaee

Partial transformations

- For sets $A, B \quad \mathbf{Set}$, we write

$$\mathbf{PT}_{AB} = \{f : f \text{ is a function } C \to B \text{ for some } C \quad A\}.$$

- Furthermore,

$$PT \;= \{(A, f, B) : A, B \quad \mathbf{Set},\; f \quad \mathbf{PT}_{AB}\} \text{ and}$$
$$PT_{AB} = \{(A, f, B) : f \quad \mathbf{PT}_{AB}\},$$

with partial product on PT , $\cdot$, defined by

$$(A, f, B) \cdot (C, g, D) = \begin{cases} (A, fg, D) & \text{if } B = C \\ \text{undefined} & \text{otherwise.} \end{cases}$$

- Finally, we define mappings

$$\lambda : PT \to \mathbf{Set} : (A, f, B) \to A \text{ and}$$
$$\rho : PT \to \mathbf{Set} : (A, f, B) \to B.$$

I. Dolinka,
I. Đurđev,
J. East,
P. Honyam,
K. Sangkhanan,
J. Sanwong,
W. Sommanaee

Green's relations and regularity in PT

Green's relations and regularity in $\mathcal{PT}$

Proposition

Let $(A, f, B), (C, g, D) \quad \mathcal{PT}$. *Then*

Green's relations and regularity in PT

Proposition

Let $(A, f, B), (C, g, D) \quad PT$. *Then*

(i) $(A, f, B) \leq_{\mathcal{R}} (C, g, D) \;\Leftrightarrow\; A = C$, $\mathrm{dom}(f) \quad \mathrm{dom}(g)$ *and* $\ker(f) \quad \ker(g)|_{\mathrm{dom}(f)}$,

Green's relations and regularity in PT

Proposition

Let $(A, f, B), (C, g, D) \quad PT$. *Then*

(i) $(A, f, B) \leq_{\mathcal{R}} (C, g, D) \;\Leftrightarrow\; A = C,\ \mathrm{dom}(f) \quad \mathrm{dom}(g)$ *and* $\ker(f) \quad \ker(g)|_{\mathrm{dom}(f)}$,

(ii) $(A, f, B) \leq_{\mathcal{L}} (C, g, D) \;\Leftrightarrow\; B = D$ *and* $\mathrm{im}(f) \quad \mathrm{im}(g)$,

Green's relations and regularity in PT

Proposition

Let $(A, f, B), (C, g, D) \quad PT$. *Then*

(i) $(A, f, B) \leq_{\mathcal{R}} (C, g, D) \;\Leftrightarrow\; A = C$, $\mathrm{dom}(f) \quad \mathrm{dom}(g)$ *and* $\ker(f) \quad \ker(g)|_{\mathrm{dom}(f)}$,

(ii) $(A, f, B) \leq_{\mathcal{L}} (C, g, D) \;\Leftrightarrow\; B = D$ *and* $\mathrm{im}(f) \quad \mathrm{im}(g)$,

(iii) $(A, f, B) \leq_{\mathcal{J}} (C, g, D) \;\Leftrightarrow\; \mathrm{rank}(f) \leq \mathrm{rank}(g)$,

Green's relations and regularity in PT

Proposition

Let $(A, f, B), (C, g, D) \quad PT$. *Then*

(i) $(A, f, B) \leq_{\mathcal{R}} (C, g, D) \;\Leftrightarrow\; A = C,\ \mathrm{dom}(f) \quad \mathrm{dom}(g)$ *and* $\ker(f) \quad \ker(g)|_{\mathrm{dom}(f)}$,

(ii) $(A, f, B) \leq_{\mathcal{L}} (C, g, D) \;\Leftrightarrow\; B = D$ *and* $\mathrm{im}(f) \quad \mathrm{im}(g)$,

(iii) $(A, f, B) \leq_{\mathcal{J}} (C, g, D) \;\Leftrightarrow\; \mathrm{rank}(f) \leq \mathrm{rank}(g)$,

(iv) $(A, f, B)\, \mathcal{R}\, (C, g, D) \;\Leftrightarrow\; A = C,\ \mathrm{dom}(f) = \mathrm{dom}(g)$ *and* $\ker(f) = \ker(g)$,

Green's relations and regularity in PT

Proposition

Let $(A,f,B),(C,g,D) \quad PT$. *Then*

(i) $(A,f,B) \leq_{\mathcal{R}} (C,g,D) \;\Leftrightarrow\; A = C,\ \mathrm{dom}(f) \quad \mathrm{dom}(g)$ *and* $\ker(f) \quad \ker(g)|_{\mathrm{dom}(f)}$,

(ii) $(A,f,B) \leq_{\mathcal{L}} (C,g,D) \;\Leftrightarrow\; B = D$ *and* $\mathrm{im}(f) \quad \mathrm{im}(g)$,

(iii) $(A,f,B) \leq_{\mathcal{J}} (C,g,D) \;\Leftrightarrow\; \mathrm{rank}(f) \leq \mathrm{rank}(g)$,

(iv) $(A,f,B)\ \mathcal{R}\ (C,g,D) \;\Leftrightarrow\; A = C,\ \mathrm{dom}(f) = \mathrm{dom}(g)$ *and* $\ker(f) = \ker(g)$,

(v) $(A,f,B)\ \mathcal{L}\ (C,g,D) \;\Leftrightarrow\; B = D$ *and* $\mathrm{im}(f) = \mathrm{im}(g)$,

Green's relations and regularity in PT

Proposition

Let $(A, f, B), (C, g, D) \quad PT$. *Then*

(i) $(A, f, B) \leq_{\mathcal{R}} (C, g, D) \;\Leftrightarrow\; A = C,\ \mathrm{dom}(f) \quad \mathrm{dom}(g)$ *and* $\ker(f) \quad \ker(g)|_{\mathrm{dom}(f)}$,

(ii) $(A, f, B) \leq_{\mathcal{L}} (C, g, D) \;\Leftrightarrow\; B = D$ *and* $\mathrm{im}(f) \quad \mathrm{im}(g)$,

(iii) $(A, f, B) \leq_{\mathcal{J}} (C, g, D) \;\Leftrightarrow\; \mathrm{rank}(f) \leq \mathrm{rank}(g)$,

(iv) $(A, f, B)\ \mathcal{R}\ (C, g, D) \;\Leftrightarrow\; A = C,\ \mathrm{dom}(f) = \mathrm{dom}(g)$ *and* $\ker(f) = \ker(g)$,

(v) $(A, f, B)\ \mathcal{L}\ (C, g, D) \;\Leftrightarrow\; B = D$ *and* $\mathrm{im}(f) = \mathrm{im}(g)$,

(vi) $(A, f, B)\ \mathcal{J}\ (C, g, D) \;\Leftrightarrow\; (A, f, B)\ \mathcal{D}\ (C, g, D) \;\Leftrightarrow\; \mathrm{rank}(f) = \mathrm{rank}(g)$.

Green's relations and regularity in PT

Proposition

Let $(A, f, B), (C, g, D) \quad PT$. *Then*

(i) $(A, f, B) \leq_{\mathcal{R}} (C, g, D) \;\Leftrightarrow\; A = C, \ \mathrm{dom}(f) \quad \mathrm{dom}(g)$ *and* $\ker(f) \quad \ker(g)|_{\mathrm{dom}(f)}$,

(ii) $(A, f, B) \leq_{\mathcal{L}} (C, g, D) \;\Leftrightarrow\; B = D$ *and* $\mathrm{im}(f) \quad \mathrm{im}(g)$,

(iii) $(A, f, B) \leq_{\mathcal{J}} (C, g, D) \;\Leftrightarrow\; \mathrm{rank}(f) \leq \mathrm{rank}(g)$,

(iv) $(A, f, B) \,\mathcal{R}\, (C, g, D) \;\Leftrightarrow\; A = C, \ \mathrm{dom}(f) = \mathrm{dom}(g)$ *and* $\ker(f) = \ker(g)$,

(v) $(A, f, B) \,\mathcal{L}\, (C, g, D) \;\Leftrightarrow\; B = D$ *and* $\mathrm{im}(f) = \mathrm{im}(g)$,

(vi) $(A, f, B) \,\mathcal{J}\, (C, g, D) \;\Leftrightarrow\; (A, f, B) \,\mathcal{D}\, (C, g, D) \;\Leftrightarrow\; \mathrm{rank}(f) = \mathrm{rank}(g)$.

Furthermore, PT *is a regular category.*

P-sets and Regularity in S_{ij}^a

P-sets and Regularity in S_{ij}^a

- $P_1^a = \{x \quad S_{ij} : xa \,\mathcal{R}\, x\}$;

P-sets and Regularity in S_{ij}^a

- $P_1^a = \{x \quad S_{ij} : xa \, \mathcal{R} \, x\}$;
- $P_2^a = \{x \quad S_{ij} : ax \, \mathcal{L} \, x\}$;

P-sets and Regularity in S_{ij}^a

- $P_1^a = \{x \quad S_{ij} : xa \, \mathcal{R} \, x\}$;
- $P_2^a = \{x \quad S_{ij} : ax \, \mathcal{L} \, x\}$;
- $P_3^a = \{x \quad S_{ij} : axa \, \mathcal{J} \, x\}$;

P-sets and Regularity in S_{ij}^a

- $P_1^a = \{x \quad S_{ij} : xa \, \mathcal{R} \, x\}$;
- $P_2^a = \{x \quad S_{ij} : ax \, \mathcal{L} \, x\}$;
- $P_3^a = \{x \quad S_{ij} : axa \, \mathcal{J} \, x\}$;
- $P^a = P_1^a \cap P_2^a$.

P-sets and Regularity in S_{ij}^a

- $P_1^a = \{x \quad S_{ij} : xa \, \mathcal{R} \, x\}$;
- $P_2^a = \{x \quad S_{ij} : ax \, \mathcal{L} \, x\}$;
- $P_3^a = \{x \quad S_{ij} : axa \, \mathcal{J} \, x\}$;
- $P^a = P_1^a \cap P_2^a$.

Proposition

Let S *be a partial semigroup, and fix* $i, j \quad I$ *and* $a \quad S_{ji}$*. Then in semigroup* S_{ij}^a *we have*

P-sets and Regularity in S_{ij}^a

- $P_1^a = \{x \in S_{ij} : xa \mathcal{R} x\}$;
- $P_2^a = \{x \in S_{ij} : ax \mathcal{L} x\}$;
- $P_3^a = \{x \in S_{ij} : axa \mathcal{J} x\}$;
- $P^a = P_1^a \cap P_2^a$.

Proposition

Let S be a partial semigroup, and fix $i, j \in I$ and $a \in S_{ji}$. Then in semigroup S_{ij}^a we have

(i) P_1^a *is a left ideal,*

P-sets and Regularity in S_{ij}^a

- $P_1^a = \{x \in S_{ij} : xa \mathcal{R} x\}$;
- $P_2^a = \{x \in S_{ij} : ax \mathcal{L} x\}$;
- $P_3^a = \{x \in S_{ij} : axa \mathcal{J} x\}$;
- $P^a = P_1^a \cap P_2^a$.

Proposition

Let S be a partial semigroup, and fix $i, j \in I$ and $a \in S_{ji}$. Then in semigroup S_{ij}^a we have

(i) P_1^a *is a left ideal,*

(ii) P_2^a *is a right ideal,*

P-sets and Regularity in S_{ij}^a

- $P_1^a = \{x \quad S_{ij} : xa \,\mathcal{R}\, x\}$;
- $P_2^a = \{x \quad S_{ij} : ax \,\mathcal{L}\, x\}$;
- $P_3^a = \{x \quad S_{ij} : axa \,\mathcal{J}\, x\}$;
- $P^a = P_1^a \cap P_2^a$.

Proposition

Let S *be a partial semigroup, and fix* $i, j \quad I$ *and* $a \quad S_{ji}$*. Then in semigroup* S_{ij}^a *we have*

(i) P_1^a *is a left ideal,*

(ii) P_2^a *is a right ideal,*

(iii) P^a *is a subsemigroup,*

P-sets and Regularity in S_{ij}^a

- $P_1^a = \{x \in S_{ij} : xa \,\mathcal{R}\, x\}$;
- $P_2^a = \{x \in S_{ij} : ax \,\mathcal{L}\, x\}$;
- $P_3^a = \{x \in S_{ij} : axa \,\mathcal{J}\, x\}$;
- $P^a = P_1^a \cap P_2^a$.

Proposition

Let S be a partial semigroup, and fix $i, j \in I$ and $a \in S_{ji}$. Then in semigroup S_{ij}^a we have

(i) P_1^a *is a left ideal,*

(ii) P_2^a *is a right ideal,*

(iii) P^a *is a subsemigroup,*

(iv) $P^a \subseteq P_3^a$,

P-sets and Regularity in S_{ij}^{a}

- $P_1^a = \{x \quad S_{ij} : xa \,\mathcal{R}\, x\}$;
- $P_2^a = \{x \quad S_{ij} : ax \,\mathcal{L}\, x\}$;
- $P_3^a = \{x \quad S_{ij} : axa \,\mathcal{J}\, x\}$;
- $P^a = P_1^a \cap P_2^a$.

Proposition

Let S be a partial semigroup, and fix $i, j \quad I$ and $a \quad S_{ji}$. Then in semigroup S_{ij}^{a} we have

(i) P_1^a *is a left ideal,*

(ii) P_2^a *is a right ideal,*

(iii) P^a *is a subsemigroup,*

(iv) $P^a \quad P_3^a$,

(v) $\mathrm{Reg}(S_{ij}^{a}) = P^a \cap \mathrm{Reg}(S)$.

P-sets and Regularity in PT_{XY}^{a}

P-sets and Regularity in PT^{a}_{XY}

X, Y **Set** with X Y, σ is an equivalence rel. on Y.

I. Dolinka, I. Đurđev, J. East, P. Honyam, K. Sangkhanan, J. Sanwong, W. Sommanaee

P-sets and Regularity in PT^{a}_{XY}

X, Y **Set** with X Y, σ is an equivalence rel. on Y.

- X *saturates* σ if each σ-class contains at least one element of X,

P-sets and Regularity in PT^a_{XY}

X, Y **Set** with X Y, σ is an equivalence rel. on Y.

- X *saturates* σ if each σ-class contains at least one element of X,
- σ *separates* X if each σ-class contains at most one element of X,

P-sets and Regularity in PT^{a}_{XY}

X, Y **Set** with X Y, σ is an equivalence rel. on Y.

- X *saturates* σ if each σ-class contains at least one element of X,
- σ *separates* X if each σ-class contains at most one element of X,

Proposition

We have

P-sets and Regularity in PT^a_{XY}

$X, Y \quad$ **Set** with $X \quad Y$, σ is an equivalence rel. on Y.

- X *saturates* σ if each σ-class contains at least one element of X,
- σ *separates* X if each σ-class contains at most one element of X,

Proposition

We have

(i) $P^a_1 = \{f \quad PT_{XY} :$
$\mathrm{im}(f) \quad \mathsf{dom}(a),\ \mathsf{ker}(a)$ *separates* $\mathrm{im}(f)\}$,

P-sets and Regularity in PT^a_{XY}

X, Y **Set** with $X \quad Y$, σ is an equivalence rel. on Y.

- X *saturates* σ if each σ-class contains at least one element of X,
- σ *separates* X if each σ-class contains at most one element of X,

Proposition

We have

(i) $P^a_1 = \{f \quad PT_{XY}$: $\mathrm{im}(f) \quad \mathrm{dom}(a)$, $\ker(a)$ *separates* $\mathrm{im}(f)\}$,

(ii) $P^a_2 = \{f \quad PT_{XY}$: $\mathrm{im}(a)$ *saturates* $\ker(f)\}$,

I. Dolinka,
I. Đurđev,
J. East,
P. Honyam,
K. Sangkhanan,
J. Sanwong,
W. Sommanaee

P-sets and Regularity in PT_{XY}^{a}

$X, Y \quad$ **Set** with $X \quad Y$, σ is an equivalence rel. on Y.

- X *saturates* σ if each σ-class contains at least one element of X,
- σ *separates* X if each σ-class contains at most one element of X,

Proposition

We have

(i) $P_1^a = \{f \quad PT_{XY} :$ $\mathrm{im}(f) \quad \mathsf{dom}(a),\ \mathsf{ker}(a)$ *separates* $\mathrm{im}(f)\}$,

(ii) $P_2^a = \{f \quad PT_{XY} : \mathrm{im}(a)$ *saturates* $\mathsf{ker}(f)\}$,

(iii) $P^a = \{f \quad PT_{XY} :$ $\mathrm{im}(f) \quad \mathsf{dom}(a),\ \mathsf{ker}(a)$ *sep* $\mathrm{im}(f),\ \mathrm{im}(a)$ *sat* $\mathsf{ker}(f)\}$,

P-sets and Regularity in PT_{XY}^{a}

X, Y **Set** with $X \quad Y$, σ is an equivalence rel. on Y.

- X *saturates* σ if each σ-class contains at least one element of X,
- σ *separates* X if each σ-class contains at most one element of X,

Proposition

We have

(i) $P_1^a = \{f \quad PT_{XY} :$ $\mathrm{im}(f) \quad \mathsf{dom}(a),\ \mathsf{ker}(a)$ *separates* $\mathrm{im}(f)\}$,

(ii) $P_2^a = \{f \quad PT_{XY} : \mathrm{im}(a)$ *saturates* $\mathsf{ker}(f)\}$,

(iii) $P^a = \{f \quad PT_{XY} :$ $\mathrm{im}(f) \quad \mathsf{dom}(a),\ \mathsf{ker}(a)$ *sep* $\mathrm{im}(f),\ \mathrm{im}(a)$ *sat* $\mathsf{ker}(f)\}$,

(iv) $P_3^a = \{f \quad PT_{XY} : \mathsf{rank}(afa) = \mathsf{rank}(f)\}$,

P-sets and Regularity in PT^a_{XY}

X, Y **Set** with $X \quad Y$, σ is an equivalence rel. on Y.

- X *saturates* σ if each σ-class contains at least one element of X,
- σ *separates* X if each σ-class contains at most one element of X,

Proposition

We have

(i) $P^a_1 = \{f \quad PT_{XY} :$ $\mathrm{im}(f) \quad \mathsf{dom}(a),\ \mathsf{ker}(a)$ *separates* $\mathrm{im}(f)\}$,

(ii) $P^a_2 = \{f \quad PT_{XY} : \mathrm{im}(a)$ *saturates* $\mathsf{ker}(f)\}$,

(iii) $P^a = \{f \quad PT_{XY} :$ $\mathrm{im}(f) \quad \mathsf{dom}(a),\ \mathsf{ker}(a)$ *sep* $\mathrm{im}(f),\ \mathrm{im}(a)$ *sat* $\mathsf{ker}(f)\}$,

(iv) $P^a_3 = \{f \quad PT_{XY} : \mathsf{rank}(afa) = \mathsf{rank}(f)\}$,

(v) $\mathsf{Reg}(PT^a_{XY}) = P^a$.

Green's relations in S_{ij}^{a}

I. Dolinka, I. Đurđev, J. East, P. Honyam, K. Sangkhanan, J. Sanwong, W. Sommanaee

Green's relations in S_{ij}^a

Theorem

Let $(S, \cdot, I, \lambda, \rho)$ *be a partial semigroup, and let* $a \quad S_{ji}$ *where* $i, j \quad I$*. If* $x \quad S_{ij}$*, then*

Green's relations in S^a_{ij}

Theorem

Let $(S, \cdot, I, \lambda, \rho)$ *be a partial semigroup, and let* $a \quad S_{ji}$ *where* $i, j \quad I$. *If* $x \quad S_{ij}$, *then*

(i) $R^a_x = \begin{cases} R_x \cap P^a_1 & \text{if } x \quad P^a_1 \\ \{x\} & \text{if } x \quad S_{ij} \setminus P^a_1, \end{cases}$

Green's relations in S^a_{ij}

Theorem

Let $(S, \cdot, I, \lambda, \rho)$ *be a partial semigroup, and let* $a \quad S_{ji}$ *where* $i, j \quad I$*. If* $x \quad S_{ij}$*, then*

(i) $R^a_x = \begin{cases} R_x \cap P^a_1 & \text{if } x \quad P^a_1 \\ \{x\} & \text{if } x \quad S_{ij} \setminus P^a_1, \end{cases}$

(ii) $L^a_x = \begin{cases} L_x \cap P^a_2 & \text{if } x \quad P^a_2 \\ \{x\} & \text{if } x \quad S_{ij} \setminus P^a_2, \end{cases}$

Green's relations in S_{ij}^a

Theorem

Let $(S, \cdot, I, \lambda, \rho)$ *be a partial semigroup, and let* $a \quad S_{ji}$ *where* $i, j \quad I$. *If* $x \quad S_{ij}$, *then*

(i) $R_x^a = \begin{cases} R_x \cap P_1^a & \text{if } x \quad P_1^a \\ \{x\} & \text{if } x \quad S_{ij} \setminus P_1^a, \end{cases}$

(ii) $L_x^a = \begin{cases} L_x \cap P_2^a & \text{if } x \quad P_2^a \\ \{x\} & \text{if } x \quad S_{ij} \setminus P_2^a, \end{cases}$

(iii) $H_x^a = \begin{cases} H_x & \text{if } x \quad P^a \\ \{x\} & \text{if } x \quad S_{ij} \setminus P^a, \end{cases}$

Green's relations in S_{ij}^a

(iv) $$D_x^a = \begin{cases} D_x \cap P^a & \text{if } x \quad P^a \\ L_x^a & \text{if } x \quad P_2^a \setminus P_1^a \\ R_x^a & \text{if } x \quad P_1^a \setminus P_2^a \\ \{x\} & \text{if } x \quad S_{ij} \setminus (P_1^a \quad P_2^a), \end{cases}$$

Green's relations in S_{ij}^a

$$(iv)\quad D_x^a = \begin{cases} D_x \cap P^a & \text{if } x \quad P^a \\ L_x^a & \text{if } x \quad P_2^a \setminus P_1^a \\ R_x^a & \text{if } x \quad P_1^a \setminus P_2^a \\ \{x\} & \text{if } x \quad S_{ij} \setminus (P_1^a \quad P_2^a), \end{cases}$$

$$(v)\quad J_x^a = \begin{cases} J_x \cap P_3^a & \text{if } x \quad P_3^a \\ D_x^a & \text{if } x \quad S_{ij} \setminus P_3^a. \end{cases}$$

Green's relations in S_{ij}^a

$$(iv)\quad D_x^a = \begin{cases} D_x \cap P^a & \text{if } x \quad P^a \\ L_x^a & \text{if } x \quad P_2^a \setminus P_1^a \\ R_x^a & \text{if } x \quad P_1^a \setminus P_2^a \\ \{x\} & \text{if } x \quad S_{ij} \setminus (P_1^a \quad P_2^a), \end{cases}$$

$$(v)\quad J_x^a = \begin{cases} J_x \cap P_3^a & \text{if } x \quad P_3^a \\ D_x^a & \text{if } x \quad S_{ij} \setminus P_3^a. \end{cases}$$

Further, if $x \quad S_{ij} \setminus P^a$, then $H_x^a = \{x\}$ is a non-group $\mathcal{H}^a$-class of S_{ij}^a.

Visually...

Visually...

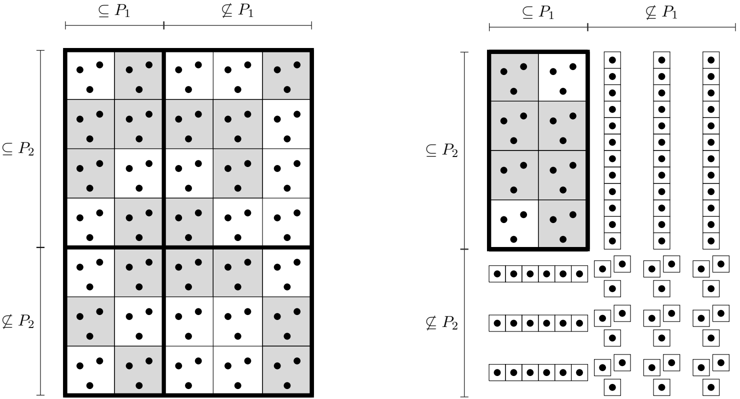


Figure: Left egg-box diagram of a $\mathcal{D}$-class in a partial semigroup Right the corresponding egg-box diagram in a sandwich semigroup

Commutative diagrams for S_{ij}^a

Commutative diagrams for S_{ij}^{a}

- $a \quad \mathrm{Reg}(S_{ji})$ is *sandwich-regular* if $aS_{ij}a \quad \mathrm{Reg}(S)$.

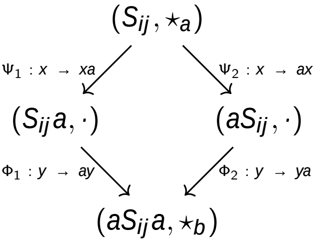
I. Dolinka,
I. Đurđev,
J. East,
P. Honyam,
K. Sangkhanan,
J. Sanwong,
W. Sommanaee

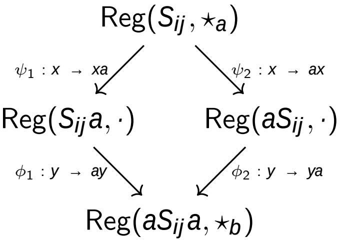
Commutative diagrams for S_{ij}^a

- $a \quad \mathrm{Reg}(S_{ji})$ is *sandwich-regular* if $aS_{ij}a \quad \mathrm{Reg}(S)$.
- Fix a sandwich-regular element $a \quad \mathrm{Reg}(S_{ji})$ and a $b \quad V(a)$.

Commutative diagrams for S_{ij}^a

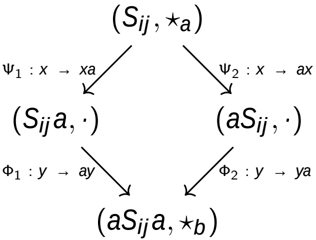
- $a \quad \mathrm{Reg}(S_{ji})$ is *sandwich-regular* if $aS_{ij}a \quad \mathrm{Reg}(S)$.
- Fix a sandwich-regular element $a \quad \mathrm{Reg}(S_{ji})$ and a $b \quad V(a)$.

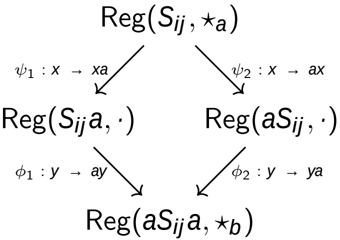




Commutative diagrams for S_{ij}^a

- a $\mathrm{Reg}(S_{ji})$ is *sandwich-regular* if $aS_{ij}a$ $\mathrm{Reg}(S)$.
- Fix a sandwich-regular element a $\mathrm{Reg}(S_{ji})$ and a b $V(a)$.



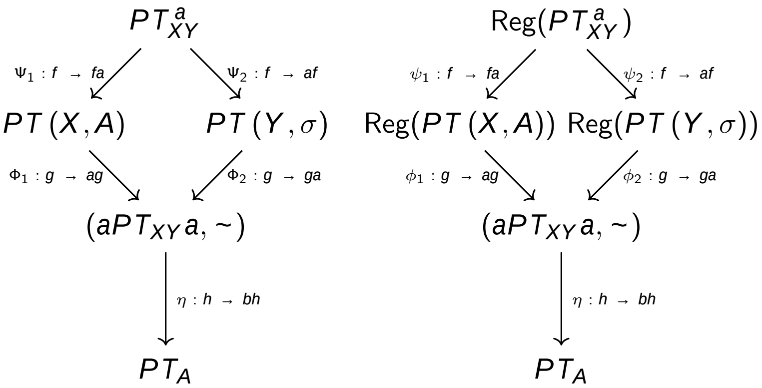


- To simplify notation, we will write

$$P^a = \mathrm{Reg}(S_{ij}, \star_a), \quad W = (aS_{ij}a, \star_b) = aP^a a,$$
$$T_1 = \mathrm{Reg}(S_{ij}a, \cdot) = P^a a, \quad T_2 = \mathrm{Reg}(aS_{ij}, \cdot) = aP^a.$$

Commutative diagrams for PT^a_{XY}

Commutative diagrams for PT^a_{XY}



Pull-back products and embedding

Pull-back products and embedding

Theorem

Let $\psi : P^a \to T_1 \times T_2 : x \to (xa, ax)$. *Then*

Pull-back products and embedding

Theorem

Let $\psi : P^a \to T_1 \times T_2 : x \to (xa, ax)$. *Then*

(i) ψ *is injective,*

I. Dolinka,
I. Đurđev,
J. East,
P. Honyam,
K. Sangkhanan,
J. Sanwong,
W. Sommanaee

Pull-back products and embedding

Theorem

Let $\psi : P^a \to T_1 \times T_2 : x \to (xa, ax)$. *Then*

(i) ψ *is injective,*

(ii) $\mathrm{im}(\psi) = \{(g, h) \quad T_1 \times T_2 : ag = ha\}$.

I. Dolinka,
I. Đurđev,
J. East,
P. Honyam,
K. Sangkhanan,
J. Sanwong,
W. Sommanaee

Pull-back products and embedding

Theorem

Let $\psi : P^a \to T_1 \times T_2 : x \to (xa, ax)$. *Then*

(i) ψ *is injective,*

(ii) $\mathrm{im}(\psi) = \{(g, h) \quad T_1 \times T_2 : ag = ha\}$.

P^a *is a pull-back product of* T_1 *and* T_2 *with respect to* W*.*

Pull-back products and embedding

Theorem

Let $\psi : P^a \to T_1 \times T_2 : x \to (xa, ax)$. *Then*

(i) ψ *is injective,*

(ii) $\mathrm{im}(\psi) = \{(g,h) \quad T_1 \times T_2 : ag = ha\}$.

P^a *is a pull-back product of* T_1 *and* T_2 *with respect to* W.

Theorem

Consider the map

Pull-back products and embedding

Theorem

Let $\psi : P^a \to T_1 \times T_2 : x \to (xa, ax)$. *Then*

(i) ψ *is injective,*

(ii) $\mathrm{im}(\psi) = \{(g,h) \quad T_1 \times T_2 : ag = ha\}$.

P^a *is a pull-back product of* T_1 *and* T_2 *with respect to* W.

Theorem

Consider the map

$$\psi : \mathrm{Reg}(PT^a_{XY}) \to \mathrm{Reg}(PT(X,A)) \times \mathrm{Reg}(PT(Y,\sigma)) : f \to (fa, af).$$

Pull-back products and embedding

Theorem

Let $\psi : P^a \to T_1 \times T_2 : x \to (xa, ax)$*. Then*

(i) ψ *is injective,*

(ii) $\mathrm{im}(\psi) = \{(g,h) \quad T_1 \times T_2 : ag = ha\}$.

P^a *is a pull-back product of* T_1 *and* T_2 *with respect to* W*.*

Theorem

Consider the map

$$\psi : \mathrm{Reg}(PT^a_{XY}) \to \mathrm{Reg}(PT(X, A)) \times \mathrm{Reg}(PT(Y, \sigma)) : f \to (fa, af).$$

Then ψ *is injective, and*

Pull-back products and embedding

Theorem

Let $\psi : P^a \to T_1 \times T_2 : x \to (xa, ax)$. *Then*

(i) ψ *is injective,*

(ii) $\mathrm{im}(\psi) = \{(g,h) \quad T_1 \times T_2 : ag = ha\}$.

P^a *is a pull-back product of* T_1 *and* T_2 *with respect to* W.

Theorem

Consider the map

$$\psi : \mathrm{Reg}(PT^a_{XY}) \to \mathrm{Reg}(PT(X,A)) \times \mathrm{Reg}(PT(Y,\sigma)) : f \to (fa, af).$$

Then ψ *is injective, and*

$$\mathrm{im}(\psi) = \{(g,h) \quad \mathrm{Reg}(PT(X,A)) \times \mathrm{Reg}(PT(Y,\sigma)) : ag = ha\}.$$

Pull-back products and embedding

Theorem

Let $\psi : P^a \to T_1 \times T_2 : x \to (xa, ax)$. *Then*

(i) ψ *is injective,*

(ii) $\mathrm{im}(\psi) = \{(g, h) \quad T_1 \times T_2 : ag = ha\}$.

P^a *is a pull-back product of* T_1 *and* T_2 *with respect to* W.

Theorem

Consider the map

$$\psi : \mathrm{Reg}(PT^a_{XY}) \to \mathrm{Reg}(PT(X, A)) \times \mathrm{Reg}(PT(Y, \sigma)) : f \to (fa, af).$$

Then ψ *is injective, and*

$$\mathrm{im}(\psi) = \{(g, h) \quad \mathrm{Reg}(PT(X, A)) \times \mathrm{Reg}(PT(Y, \sigma)) : ag = ha\}.$$

$\mathrm{Reg}(PT^a_{XY})$ *is a pull-back product of* $\mathrm{Reg}(PT(X, A))$ *and* $\mathrm{Reg}(PT(Y, \sigma))$ *with respect to* PT_A.

2

The regular subsemigroup

The regular subsemigroup

Lemma
If a *is sandwich-regular, then* $P^a = \mathrm{Reg}(S_{ij}^a)$.

The regular subsemigroup

Lemma

If a *is sandwich-regular, then* $P^a = \mathrm{Reg}(S^a_{ij})$.

Lemma

Suppose T *is a semigroup for which* $Q = \mathrm{Reg}(T)$ *is a subsemigroup of* T. *If* $\mathcal{K}$ *is any of Green's relations on* T *other than* $\mathcal{J}$, *then* $\mathcal{K}^Q = \mathcal{K} \cap (Q \times Q)$.

The regular subsemigroup

Lemma
If a is sandwich-regular, then $P^a = \mathrm{Reg}(S_{ij}^a)$.

Lemma
Suppose T is a semigroup for which $Q = \mathrm{Reg}(T)$ is a subsemigroup of T. If $\mathcal{K}$ is any of Green's relations on T other than $\mathcal{J}$, then $\mathcal{K}^Q = \mathcal{K} \cap (Q \times Q)$.

- For $x \quad P^a$ denote

$$\overline{x} = x\phi = axa \quad W,$$

An example: PT_{35}^{a}

An example: PT_{35}^{a}

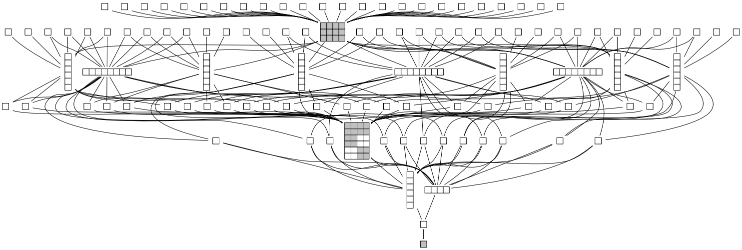


Figure: Egg-box diagram of PT_{35}^{a}, where $a = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 2 & 2 & - \end{pmatrix}$ PT_{53}.
Note that a is neither full, nor injective, nor surjective.

An example: PT_{35}^{a}

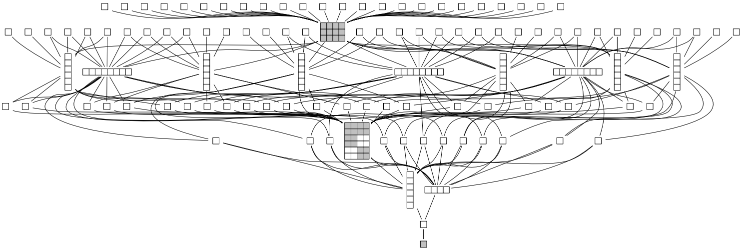


Figure: Egg-box diagram of PT_{35}^{a}, where $a = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 2 & 2 & - \end{pmatrix} \quad PT_{53}$. Note that a is neither full, nor injective, nor surjective.

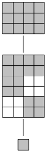


Figure: Egg-box diagram of $\mathsf{Reg}(PT_{35}^{a})$. It is an "inflation" of PT_2

Description of the inflation

Description of the inflation

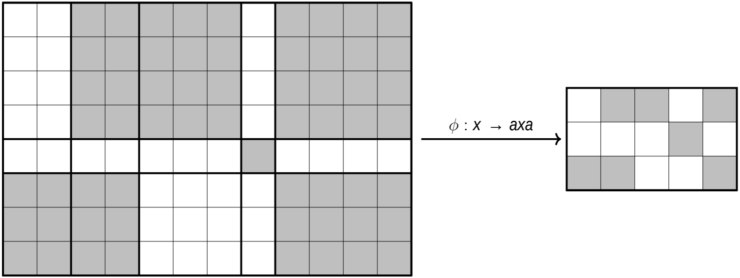


Figure: A $\mathcal{D}^a$-class of P^a (left) and its corresponding $\mathcal{D}^{\sim}$-class of W (right)

.

Idempotents and the Local Monoid Covering property

Idempotents and the Local Monoid Covering property

Lemma

We have $E_a(P^a) = E_a(S^a_{ij}) = E_b(W)\phi^{-1}$.

Idempotents and the Local Monoid Covering property

Lemma

We have $E_a(P^a) = E_a(S^a_{ij}) = E_b(W)\phi^{-1}$.

- We write $\mathsf{E}_a(P^a) = \ E_a(P^a)\ _a$ and $\mathsf{E}_b(W) = \ E_b(W)\ _b$ for the idempotent-generated subsemigroups.

Idempotents and the Local Monoid Covering property

Lemma

We have $E_a(P^a) = E_a(S^a_{ij}) = E_b(W)\phi^{-1}$.

- We write $\mathsf{E}_a(P^a) = \ E_a(P^a) \ _a$ and $\mathsf{E}_b(W) = \ E_b(W) \ _b$ for the idempotent-generated subsemigroups.

Theorem

We have $\mathsf{E}_a(S^a_{ij}) = \mathsf{E}_a(P^a) = \mathsf{E}_b(W)\phi^{-1}$.

Idempotents and the Local Monoid Covering property

Lemma

We have $E_a(P^a) = E_a(S^a_{ij}) = E_b(W)\phi^{-1}$.

- We write $\mathsf{E}_a(P^a) = \ E_a(P^a)\ _a$ and $\mathsf{E}_b(W) = \ E_b(W)\ _b$ for the idempotent-generated subsemigroups.

Theorem

We have $\mathsf{E}_a(S^a_{ij}) = \mathsf{E}_a(P^a) = \mathsf{E}_b(W)\phi^{-1}$.

- For $e, f \quad E(T)$ we define $e \quad f$ if $efe = f$.

Idempotents and the Local Monoid Covering property

Lemma

We have $E_a(P^a) = E_a(S^a_{ij}) = E_b(W)\phi^{-1}$.

- We write $\mathsf{E}_a(P^a) = \ E_a(P^a)\ _a$ and $\mathsf{E}_b(W) = \ E_b(W)\ _b$ for the idempotent-generated subsemigroups.

Theorem

We have $\mathsf{E}_a(S^a_{ij}) = \mathsf{E}_a(P^a) = \mathsf{E}_b(W)\phi^{-1}$.

- For $e, f \quad E(T)$ we define $e \quad f$ if $efe = f$.
- We say that P^a has the *Local Monoid Covering* (LMC) property if every idempotent of P^a is   -below an element of $V(a)$.

Corollaries of the LMC property

Corollaries of the LMC property

Proposition

Let $r = |\widehat{H}^a_b/\mathcal{R}^a|$ *and* $l = |\widehat{H}^a_b/\mathcal{L}^a|$, *let* M *be a full submonoid of* W *for which* $M \setminus G_M$ *is an ideal of* M, *and put* $N = M\phi^{-1}$. *Then*

$$\mathrm{rank}(N) \geq \mathrm{rank}(M : G_M) + \max(r, l, \mathrm{rank}(G_M)),$$

with equality if P^a *has the LMC property.*

Corollaries of the LMC property

Proposition

Let $r = |\widehat{H}^a_b/\mathcal{R}^a|$ *and* $l = |\widehat{H}^a_b/\mathcal{L}^a|$, *let* M *be a full submonoid of* W *for which* $M \setminus G_M$ *is an ideal of* M, *and put* $N = M\phi^{-1}$. *Then*

$$\mathrm{rank}(N) \geq \mathrm{rank}(M : G_M) + \max(r, l, \mathrm{rank}(G_M)),$$

with equality if P^a *has the LMC property.*

Theorem

Let $r = |\widehat{H}^a_b/\mathcal{R}^a|$ *and* $l = |\widehat{H}^a_b/\mathcal{L}^a|$. *Then*

$$\mathrm{rank}(\mathsf{E}_a(P^a)) \geq \mathrm{rank}(\mathsf{E}_b(W)) + \max(r, l) - 1$$
$$\mathrm{idrank}(\mathsf{E}_a(P^a)) \geq \mathrm{idrank}(\mathsf{E}_b(W)) + \max(r, l) - 1.$$

with equality in both if P^a *has the LMC property.*

Ranks in PT^a_{XY}

Ranks in PT^a_{XY}

Theorem

(i) *If* $|P^a| \geq {}_0$, *then* $\mathrm{rank}(P^a) = |P^a|$.

(ii) *If* $|P^a| < {}_0$, *then*

$$\mathrm{rank}(P^a) = \begin{cases} 1 & \text{if } \alpha = 0 \\ 1 + \max(2^\beta, \Lambda_I) & \text{if } \alpha = 1 \\ 2 + \max(3^\beta, \Lambda_I) & \text{if } \alpha = 2 \\ 2 + \max((\alpha+1)^\beta, \Lambda_I, 2) & \text{if } \alpha \geq 3. \end{cases}$$

Ranks in PT_{XY}^{a}

Theorem

(i) *If* $|P^a| \geq {}_0$*, then* $\operatorname{rank}(P^a) = |P^a|$.

(ii) *If* $|P^a| < {}_0$*, then*

$$\operatorname{rank}(P^a) = \begin{cases} 1 & \textit{if } \alpha = 0 \\ 1 + \max(2^\beta, \Lambda_I) & \textit{if } \alpha = 1 \\ 2 + \max(3^\beta, \Lambda_I) & \textit{if } \alpha = 2 \\ 2 + \max((\alpha+1)^\beta, \Lambda_I, 2) & \textit{if } \alpha \geq 3. \end{cases}$$

Theorem

We have $E_{XY}^{a} = \mathsf{E}_a(PT_{XY}^{a}) = \mathsf{E}(PT_A)\varphi^{-1}$*. Further,* $\operatorname{rank}(E_{XY}^{a}) = \operatorname{idrank}(E_{XY}^{a})$ *and*

$$\operatorname{rank}(E_{XY}^{a}) = \begin{cases} |E_{XY}^{a}| = |P^a| & \textit{if } |P^a| \geq {}_0 \\ \binom{\alpha+1}{2} + \max\left((\alpha+1)^\beta, \Lambda_I\right) & \textit{if } |P^a| < {}_0. \end{cases}$$

Ranks in PT^a_{XY}

Theorem

(i) *If* $|P^a| \geq \ _0$*, then* $\mathrm{rank}(P^a) = |P^a|$.

(ii) *If* $|P^a| < \ _0$*, then*

$$\mathrm{rank}(P^a) = \begin{cases} 1 & \text{if } \alpha = 0 \\ 1 + \max(2^\beta, \Lambda_I) & \text{if } \alpha = 1 \\ 2 + \max(3^\beta, \Lambda_I) & \text{if } \alpha = 2 \\ 2 + \max((\alpha+1)^\beta, \Lambda_I, 2) & \text{if } \alpha \geq 3. \end{cases}$$

Theorem

We have $E^a_{XY} = \mathsf{E}_a(PT^a_{XY}) = \mathsf{E}(PT_A)\varphi^{-1}$. *Further,* $\mathrm{rank}(E^a_{XY}) = \mathrm{idrank}(E^a_{XY})$ *and*

$$\mathrm{rank}(E^a_{XY}) = \begin{cases} |E^a_{XY}| = |P^a| & \text{if } |P^a| \geq \ _0 \\ \binom{\alpha+1}{2} + \max\left((\alpha+1)^\beta, \Lambda_I\right) & \text{if } |P^a| < \ _0. \end{cases}$$

- $\mathrm{rank}(PT^a_{XY})$ depends on a. (Injective? Surjective? Full?)

Further research

Further research

- Sandwich semigroups of "special" types of transformation semigroups (e.g. Order-preserving),

Further research

- Sandwich semigroups of "special" types of transformation semigroups (e.g. Order-preserving),
- sandwich semigroups of diagrams,

Further research

- Sandwich semigroups of "special" types of transformation semigroups (e.g. Order-preserving),
- sandwich semigroups of diagrams,
- connections to non-sandwich semigroups. . .

Thank you for your attention!