Sandwich semigroups in locally small categories

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Outline

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- $\mathcal{S}^a_{XY} = (\mathcal{S}_{XY}, \star_a)$ is a semigroup, where

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Furthermore, $\mathrm{Reg}(S) = \{x \in S : x = xyx\ (\exists y \in S)\}$.

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Green's relations and regularity in $\mathcal{PT}$

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Furthermore, $\mathcal{PT}$ *is a regular category.*

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(iv) $P^a \subseteq P_3^a$,
(v) $\mathrm{Reg}(S_{ij}^a) = P^a \cap \mathrm{Reg}(S)$.

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P-sets and Regularity in $\mathcal{PT}^a_{XY}$

$X, Y \in \mathbf{Set}$ with $X \subseteq Y$, σ is an equivalence rel. on Y.

- X *saturates* σ if each σ-class contains at least one element of X,
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(v) $\mathsf{Reg}(\mathcal{PT}^a_{XY}) = P^a$.

Green's relations in S_{ij}^a

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Theorem

Let $(S, \cdot, I, \lambda, \rho)$ *be a partial semigroup, and let* $a \in S_{ji}$ *where* $i, j \in I$*. If* $x \in S_{ij}$*, then*

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Green's relations in S_{ij}^a

(iv) $$D_x^a = \begin{cases} D_x \cap P^a & \text{if } x \in P^a \\ L_x^a & \text{if } x \in P_2^a \setminus P_1^a \\ R_x^a & \text{if } x \in P_1^a \setminus P_2^a \\ \{x\} & \text{if } x \in S_{ij} \setminus (P_1^a \cup P_2^a), \end{cases}$$

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Further, if $x \in S_{ij} \setminus P^a$, then $H_x^a = \{x\}$ is a non-group $\mathcal{H}^a$-class of S_{ij}^a.

Visually...

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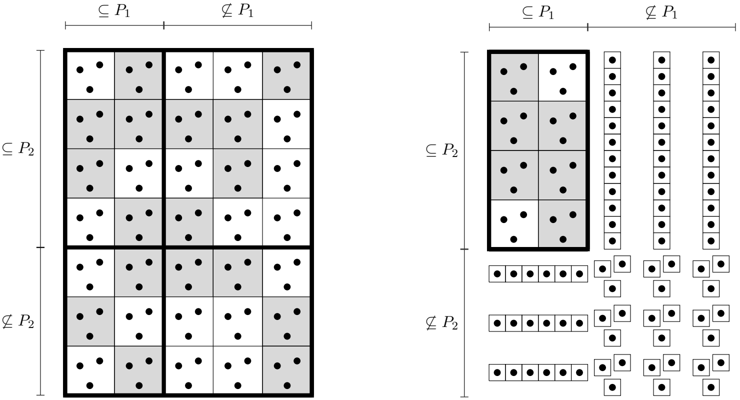

Figure: Left egg-box diagram of a $\mathcal{D}$-class in a partial semigroup Right the corresponding egg-box diagram in a sandwich semigroup

Commutative diagrams for S_{ij}^a

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- $a \in \mathsf{Reg}(S_{ji})$ is *sandwich-regular* if $aS_{ij}a \subseteq \mathsf{Reg}(S)$.

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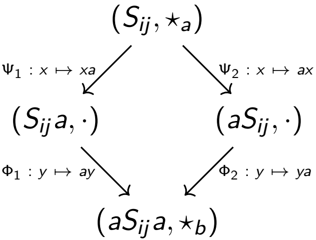

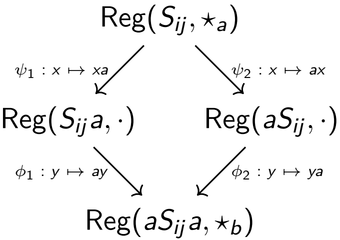

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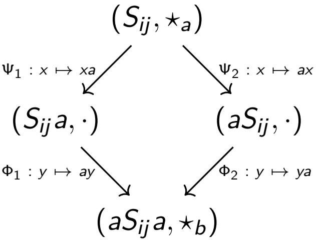

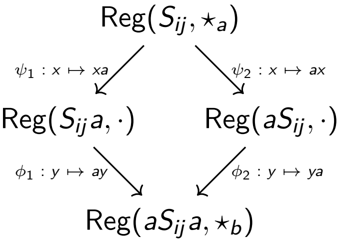

- To simplify notation, we will write

$$P^a = \mathrm{Reg}(S_{ij}, \star_a), \qquad W = (aS_{ij}a, \star_b) = aP^a a,$$
$$T_1 = \mathrm{Reg}(S_{ij}a, \cdot) = P^a a, \qquad T_2 = \mathrm{Reg}(aS_{ij}, \cdot) = aP^a.$$

Commutative diagrams for $\mathcal{PT}^a_{XY}$

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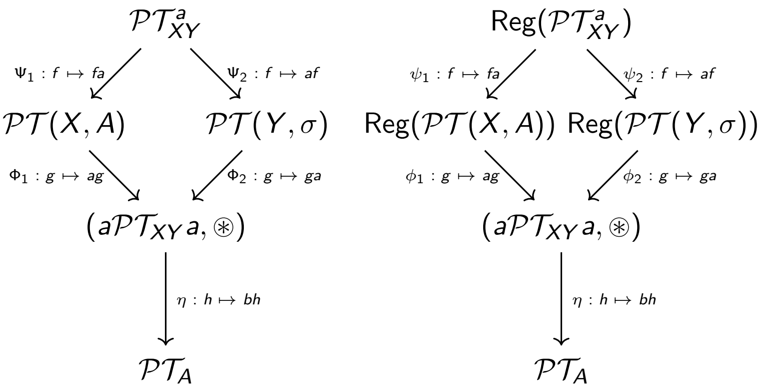

Pull-back products and embedding

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Theorem

Let $\psi : P^a \to T_1 \times T_2 : x \mapsto (xa, ax)$. *Then*

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Consider the map

$$\psi : \mathrm{Reg}(\mathcal{PT}^a_{XY}) \to \mathrm{Reg}(\mathcal{PT}(X, A)) \times \mathrm{Reg}(\mathcal{PT}(Y, \sigma)) : f \mapsto (fa, af).$$

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- For $x \in P^a$ denote

$$\overline{x} = x\phi = axa \in W,$$

An example: $\mathcal{PT}_{35}^{a}$

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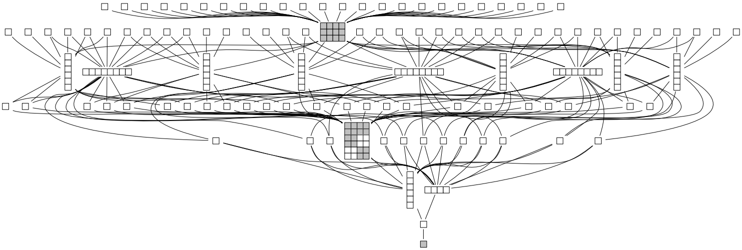

Figure: Egg-box diagram of $\mathcal{PT}_{35}^{a}$, where $a = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 2 & 2 & - \end{pmatrix} \in \mathcal{PT}_{53}$. Note that a is neither full, nor injective, nor surjective.

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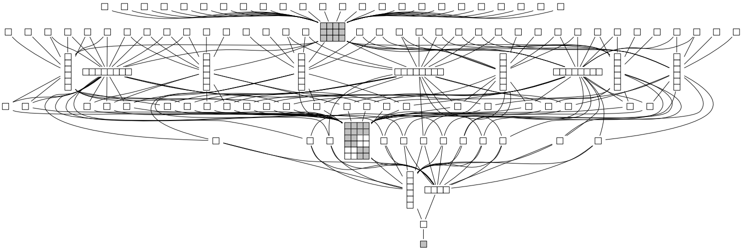

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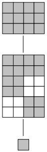

Figure: Egg-box diagram of $\mathsf{Reg}(\mathcal{PT}_{35}^{a})$. It is an "inflation" of $\mathcal{PT}_2$

Description of the inflation

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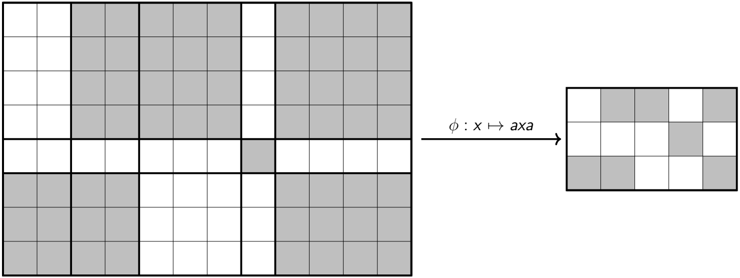

Figure: A $\mathcal{D}^a$-class of P^a (left) and its corresponding $\mathcal{D}^{\circledast}$-class of W (right)

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Idempotents and the Local Monoid Covering property

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- We write $\mathbb{E}_a(P^a) = \langle E_a(P^a)\rangle_a$ and $\mathbb{E}_b(W) = \langle E_b(W)\rangle_b$ for the idempotent-generated subsemigroups.

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- For $e, f \in E(T)$ we define $e \preceq f$ if $efe = f$.
- We say that P^a has the *Local Monoid Covering* (LMC) property if every idempotent of P^a is $\preceq$-below an element of $V(a)$.

Corollaries of the LMC property

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Proposition

Let $r = |\widehat{H}_b^a/\mathcal{R}^a|$ *and* $l = |\widehat{H}_b^a/\mathcal{L}^a|$, *let* M *be a full submonoid of* W *for which* $M \setminus G_M$ *is an ideal of* M, *and put* $N = M\phi^{-1}$. *Then*

$$\mathrm{rank}(N) \geq \mathrm{rank}(M : G_M) + \max(r, l, \mathrm{rank}(G_M)),$$

with equality if P^a *has the LMC property.*

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$$\mathrm{rank}(\mathbb{E}_a(P^a)) \geq \mathrm{rank}(\mathbb{E}_b(W)) + \max(r, l) - 1$$
$$\mathrm{idrank}(\mathbb{E}_a(P^a)) \geq \mathrm{idrank}(\mathbb{E}_b(W)) + \max(r, l) - 1.$$

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Ranks in $\mathcal{PT}^a_{XY}$

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Theorem

We have $\mathcal{E}^a_{XY} = \mathbb{E}_a(\mathcal{PT}^a_{XY}) = \mathbb{E}(\mathcal{PT}_A)\varphi^{-1}$. *Further,* $\mathrm{rank}(\mathcal{E}^a_{XY}) = \mathrm{idrank}(\mathcal{E}^a_{XY})$ *and*

$$\mathrm{rank}(\mathcal{E}^a_{XY}) = \begin{cases} |\mathcal{E}^a_{XY}| = |P^a| & \text{if } |P^a| \geq \aleph_0 \\ \binom{\alpha+1}{2} + \max\left((\alpha+1)^\beta, \Lambda_I\right) & \text{if } |P^a| < \aleph_0. \end{cases}$$

Ranks in $\mathcal{PT}^a_{XY}$

Theorem

(i) *If* $|P^a| \geq \aleph_0$, *then* $\mathrm{rank}(P^a) = |P^a|$.

(ii) *If* $|P^a| < \aleph_0$, *then*

$$\mathrm{rank}(P^a) = \begin{cases} 1 & \text{if } \alpha = 0 \\ 1 + \max(2^\beta, \Lambda_I) & \text{if } \alpha = 1 \\ 2 + \max(3^\beta, \Lambda_I) & \text{if } \alpha = 2 \\ 2 + \max((\alpha+1)^\beta, \Lambda_I, 2) & \text{if } \alpha \geq 3. \end{cases}$$

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- $\mathrm{rank}(\mathcal{PT}^a_{XY})$ depends on a. (Injective? Surjective? Full?)

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- connections to non-sandwich semigroups...

Thank you for your attention!