Algebras of incidence structures: representing regular double p-algebras

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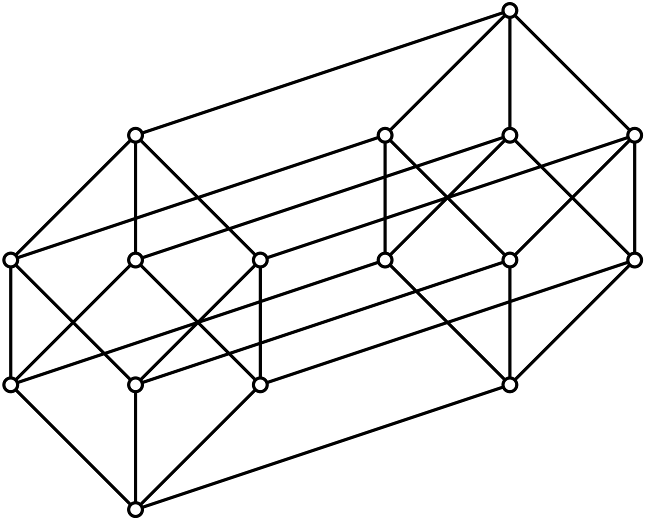

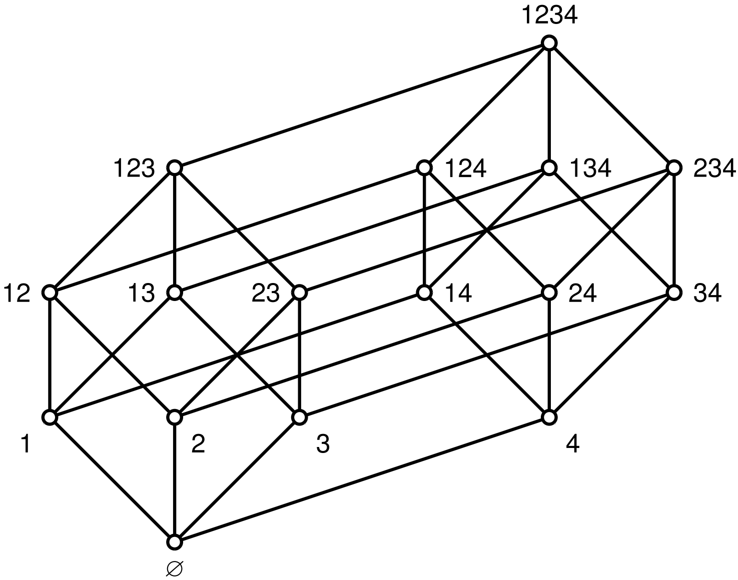
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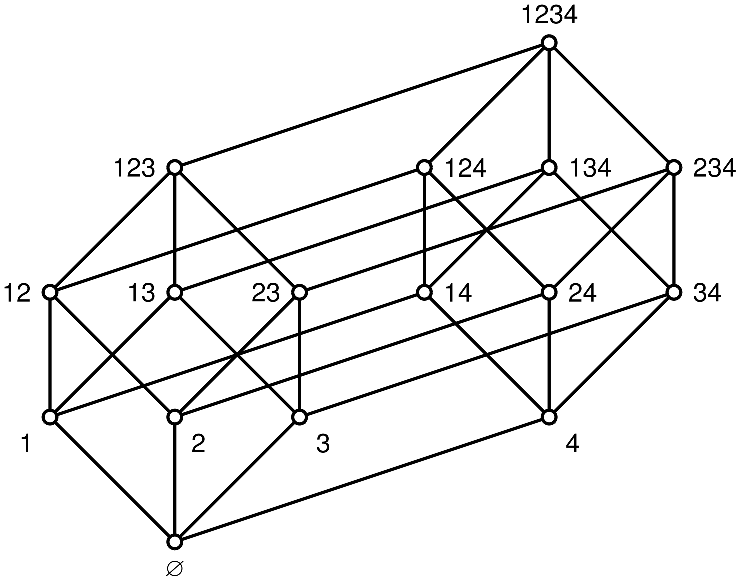

Fact: Every finite boolean lattice is isomorphic to a powerset lattice.

A graph:

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A subgraph:

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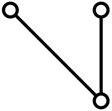

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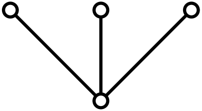

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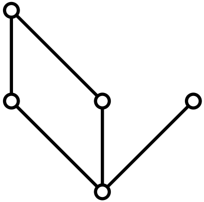

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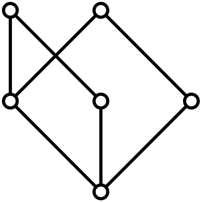

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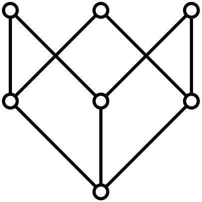

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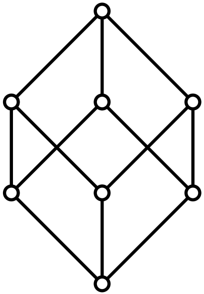

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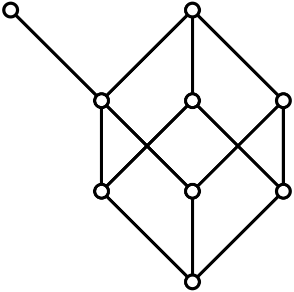

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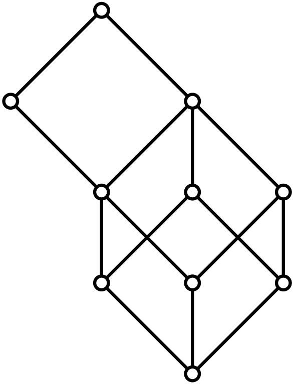

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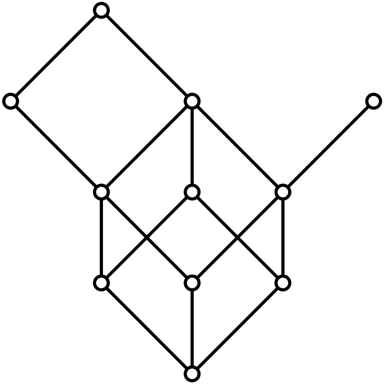

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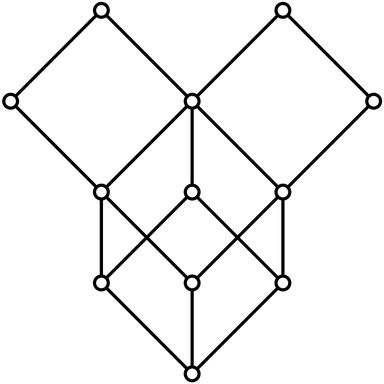

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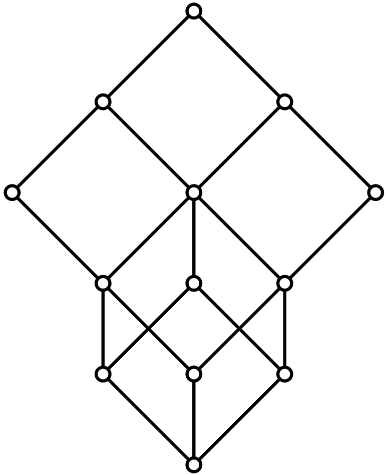

Let $G = \langle V, E \rangle$ be a graph. The set of all subgraphs[1] of G, ordered by inclusion, is a bounded distributive lattice, where

$$\langle V_1, E_1 \rangle \vee \langle V_2, E_2 \rangle = \langle V_1 \cup V_2, E_1 \cup E_2 \rangle$$
$$\langle V_1, E_1 \rangle \wedge \langle V_2, E_2 \rangle = \langle V_1 \cap V_2, E_1 \cap E_2 \rangle.$$

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Proposition (Reyes & Zolfaghari, 1996)

Let G be a graph. The lattice $\mathcal{S}(G)$ forms a double-Heyting algebra.

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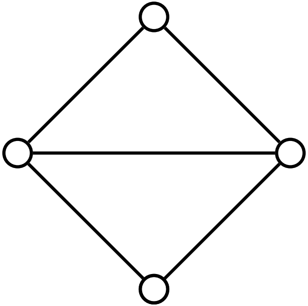

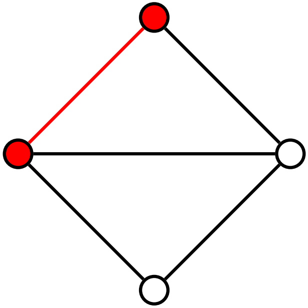

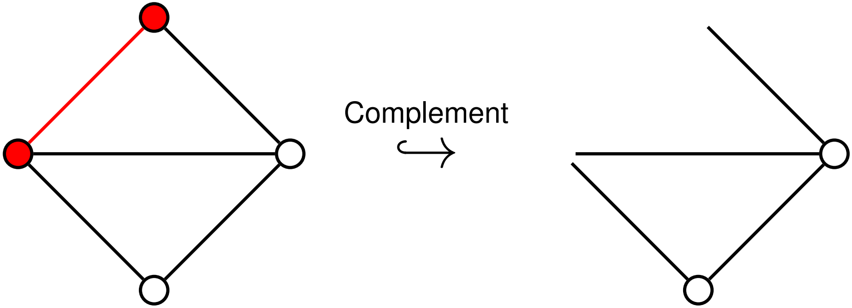
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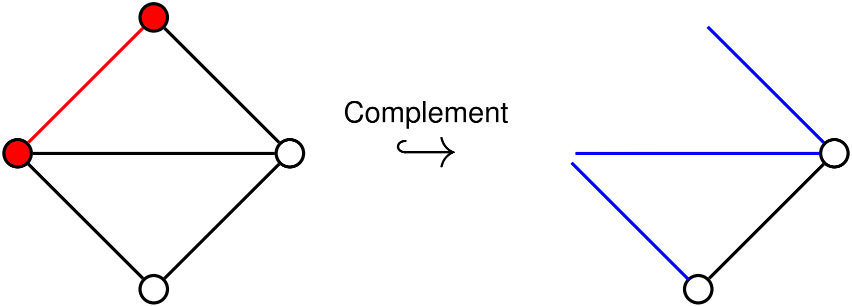

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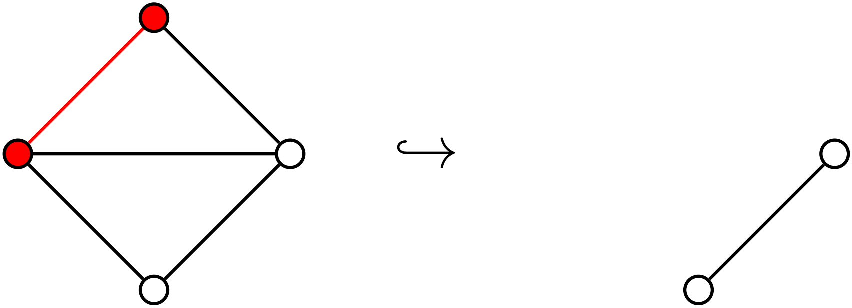

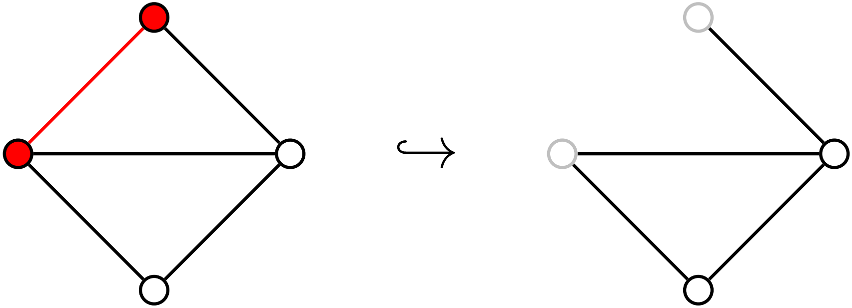

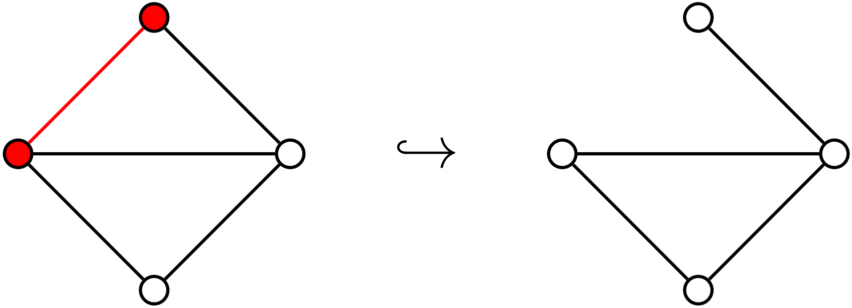

A unary operation $\neg$ on a bounded lattice L is a *pseudocomplement* operation if, for every $x \in L$, there exists an element $\neg x \in L$ such that

$$x \wedge y = 0 \iff y \leq \neg x.$$

That is, $\neg x$ is the largest element $y \in L$ such that $x \wedge y = 0$.

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A *double p-algebra* is a bounded lattice with its signature enriched by these two operations.

Pseudocomplements of subgraphs

Take the set complement of the subgraph and abandon the extra edges. Formally, for a graph $G = \langle V, E \rangle$ and a subgraph $H = \langle V', E' \rangle$,

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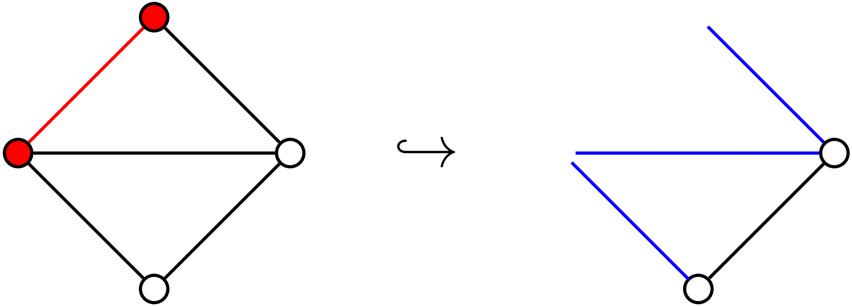

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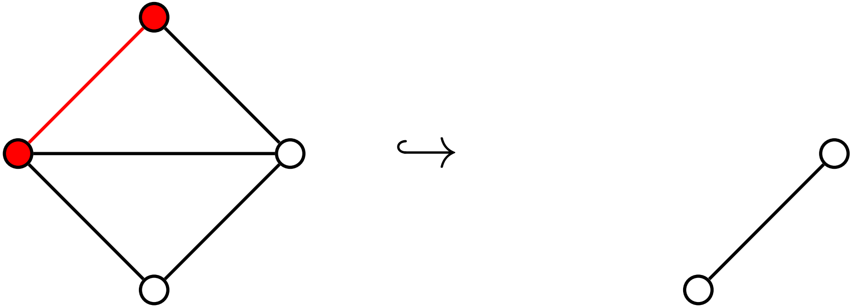

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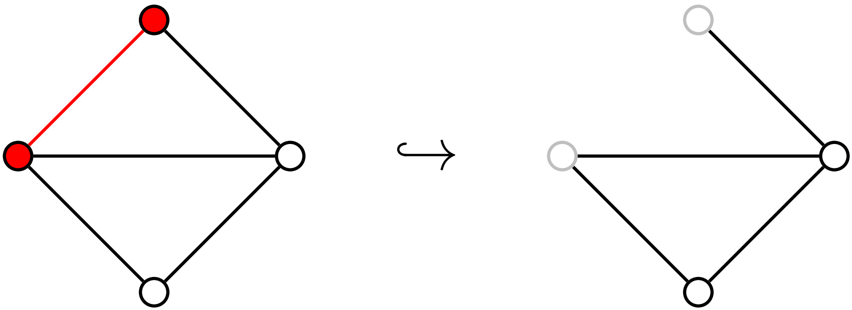

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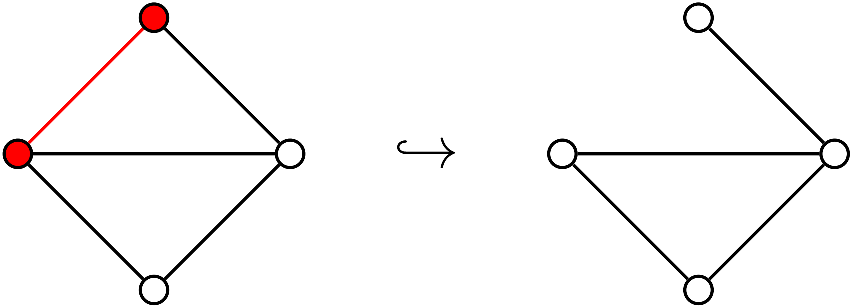

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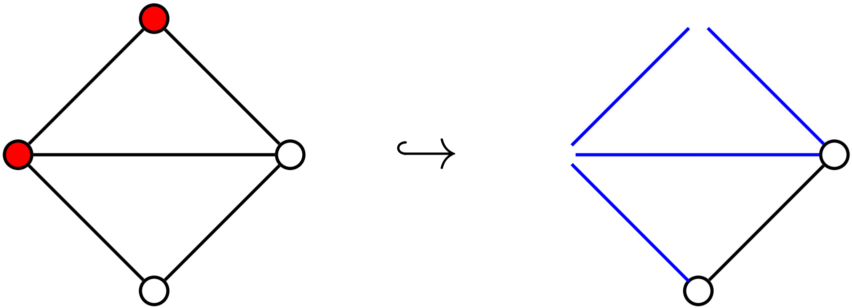

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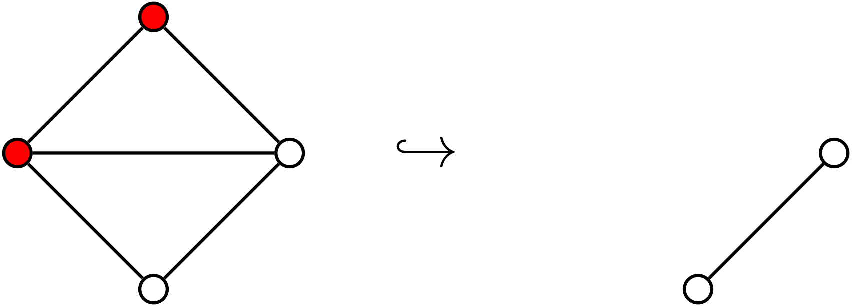

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Theorem

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If $\neg A = \neg B$ then $V \setminus V_A = V \setminus V_B$, so they have the same vertices. Similarly, if $\sim A = \sim B$ then they have the same edges. □

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Let **A** *be a double p-algebra.*

- *(Varlet, 1972) The following are equivalent.*
  1. **A** *is congruence regular.*
  2. $(\forall a, b \in A)$ *if* $\neg a = \neg b$ *and* $\sim a = \sim b$ *then* $a = b$.
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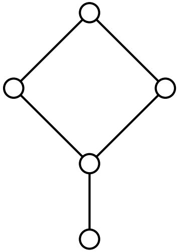
  3. $(\forall a, b \in A)$ $a \wedge \sim a \leq b \vee \neg b$.
- *(Katriňák, 1973) If* **A** *is regular then* **A** *is term-equivalent to a double-Heyting algebra via the term*

$$x \to y = \neg\neg(\neg x \vee \neg\neg y) \wedge [\sim(x \vee \neg x) \vee \neg x \vee y \vee \neg y],$$

and its dual.

Are graphs enough?



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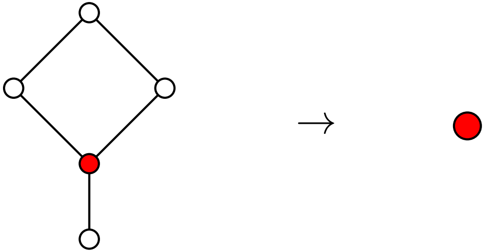

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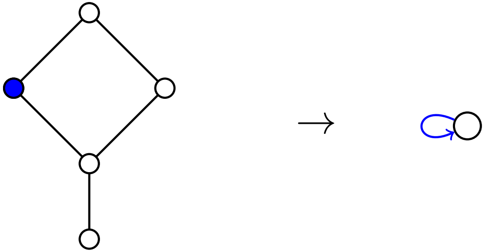

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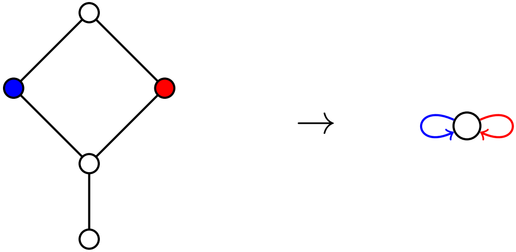

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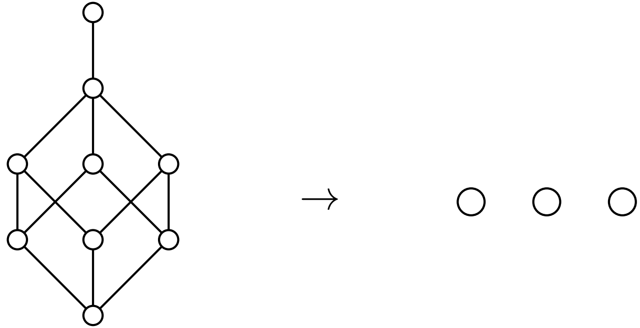

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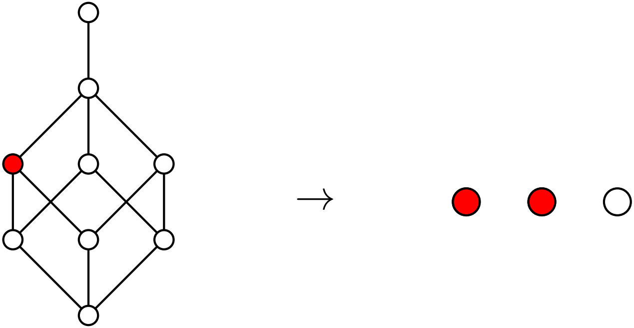

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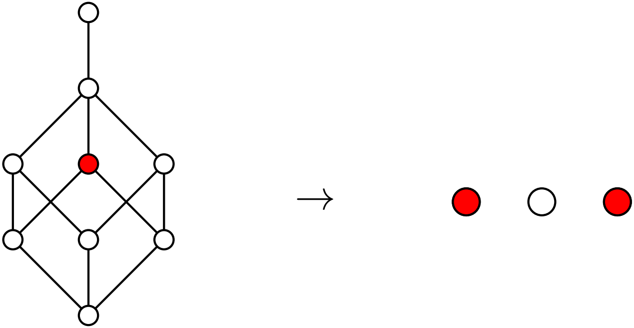

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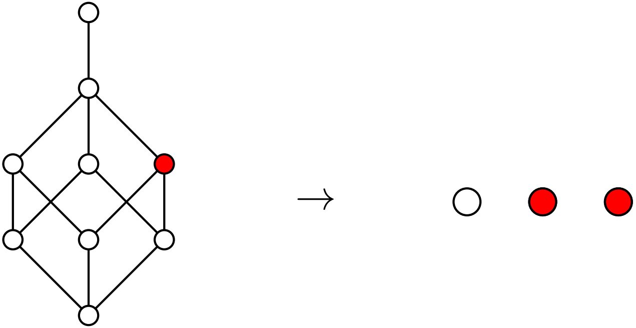

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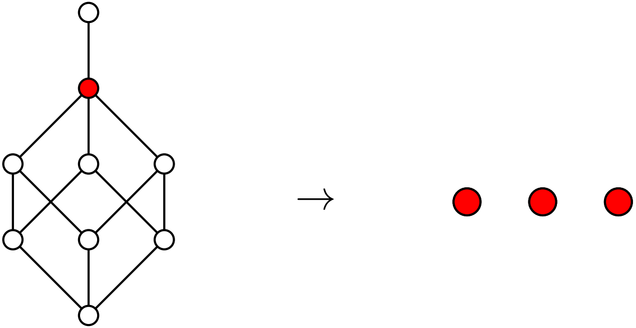

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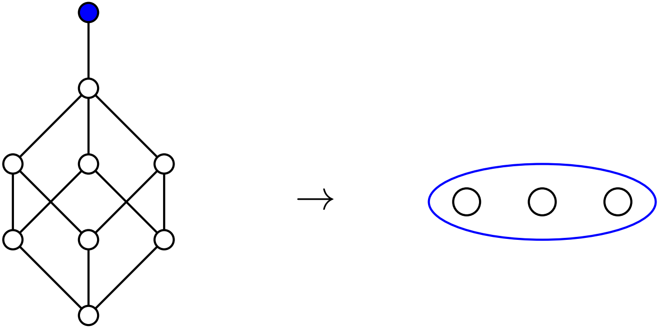

Definition

An *incidence structure*[a] is a triple $\langle P, L, I\rangle$ where P is a set of points, L is a set of lines and $I \subseteq P \times L$ is an incidence relation describing which points are incident to which lines.

[a]also known as a formal context

Example

Let $P = \{1, 2, 3\}$, $L = \{x, y, z, a, b\}$, and let

$$\begin{aligned} I = \{1,2,3\} &\times \{x, y\} \\ \cup \{1,2\} &\times \{z\} \\ \cup \{(1,a)&,(1,b)\} \end{aligned}$$

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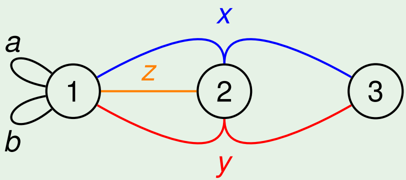

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Let $G = \langle P, L, I \rangle$ be an incidence structure. A *point-preserving substructure* of G is a pair $\langle P', L' \rangle$ such that

1. $P' \subseteq P$ and $L' \subseteq L$,
2. for all $\ell \in L'$, if $(p, \ell) \in I$ then $p \in P'$.

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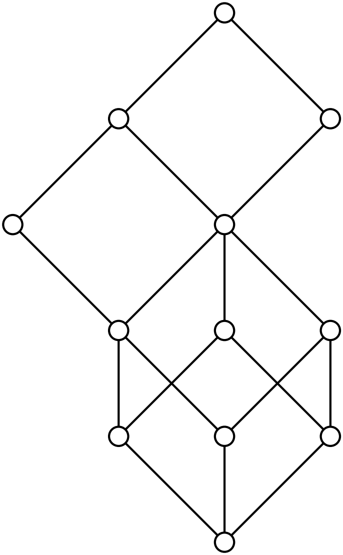
The incidence relation is defined implicitly from G.

Let $\mathcal{S}(G)$ denote the set of all point-preserving substructures of a structure G. This induces a double p-algebra in a similar way to graphs, where

$$\neg\langle P', L' \rangle = \langle P \backslash P', \{\ell \in L \backslash L' \mid (\forall p \in P)\ (p, \ell) \in I \implies p \in P \backslash P'\} \rangle$$
$$\sim\langle P', L' \rangle = \langle P \backslash P' \cup \{p \in P \mid (\exists \ell \in L \backslash L')\ (p, \ell) \in I\}, L \backslash L' \rangle.$$

Illustrating the converse



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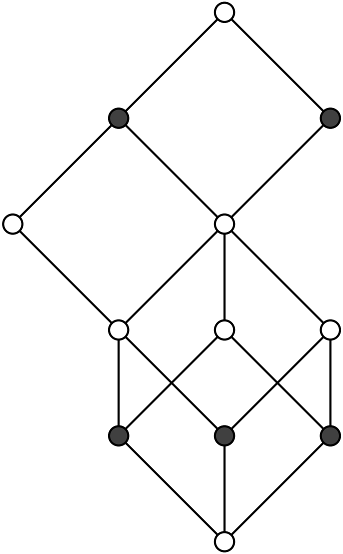

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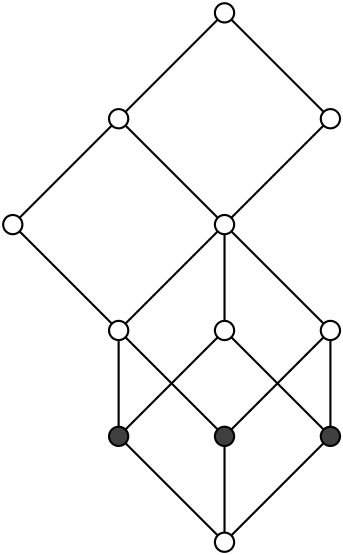

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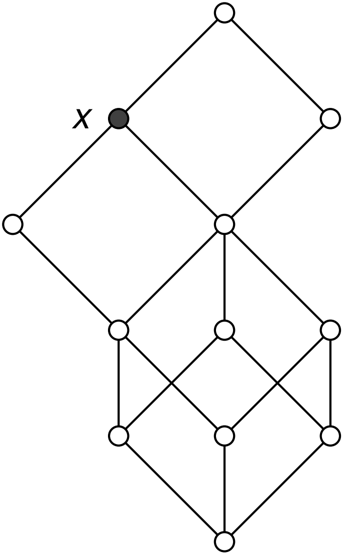

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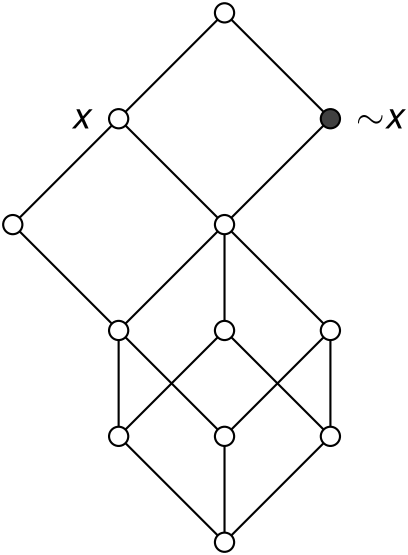

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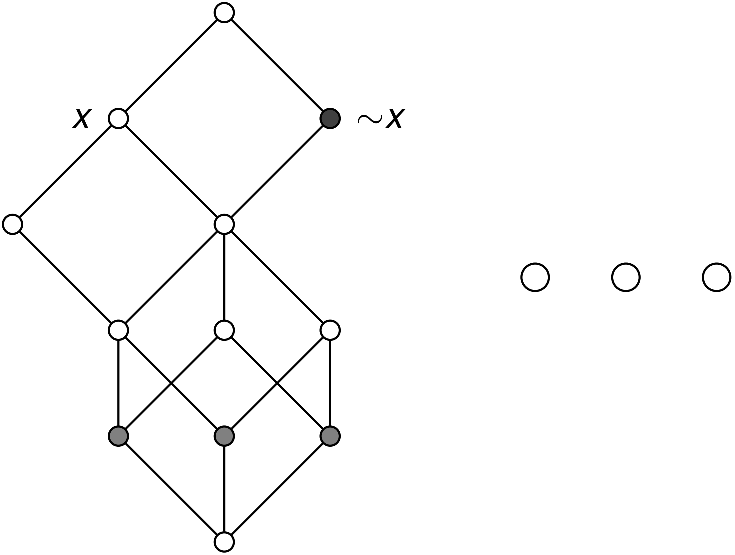

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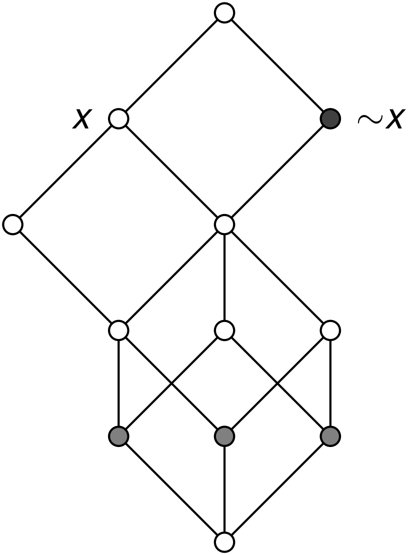

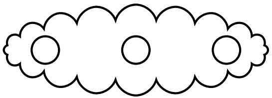

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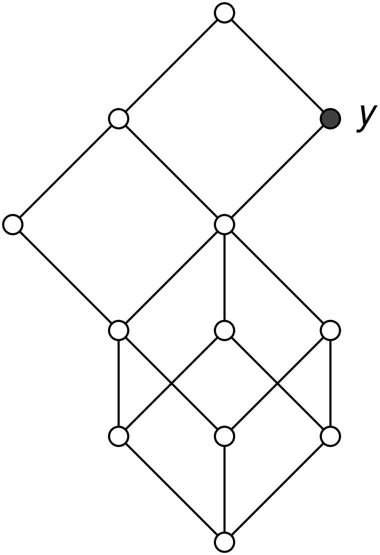

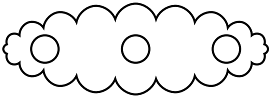

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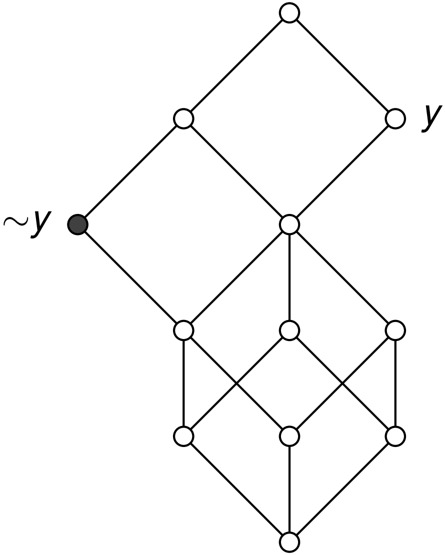

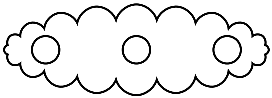

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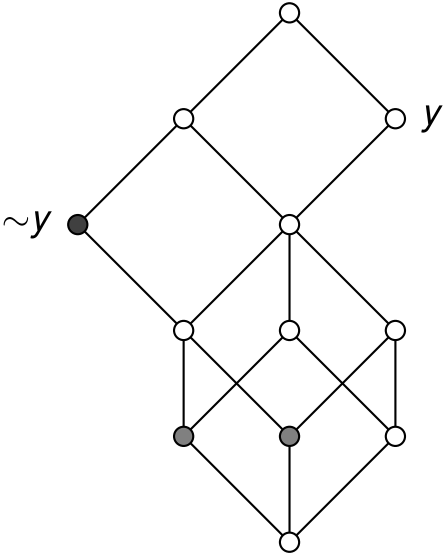

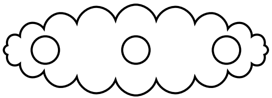

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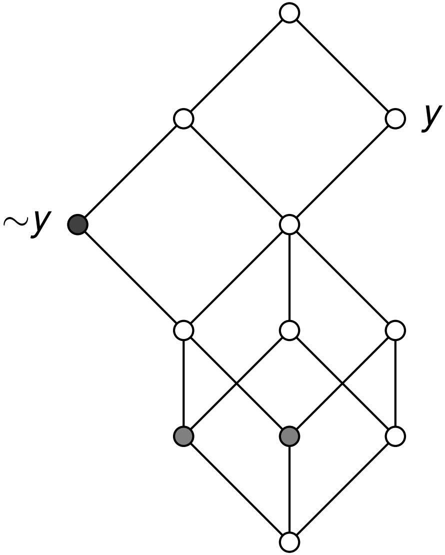

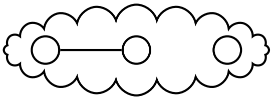

The representation theorem

Theorem (T., 2015)

Let **A** *be a regular double p-algebra. Then the following are equivalent.*

1. **A** $\cong \mathcal{P}(B) \times \mathcal{S}(G)$ *for some set* B *and some incidence structure* G*.*
2. **A** $\cong \mathcal{S}(G)$ *for some incidence structure* G*.*
3. **A** *is complete, completely distributive and doubly atomic.*

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Corollary

Every finite regular double p-algebra is isomorphic to $\mathcal{S}(G)$ *for some incidence structure G.*

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Let **A** *be a regular double p-algebra. Then there is an incidence structure* G *such that* **A** *embeds into* $\mathcal{S}(G)$*.*