Taylor and Malt'sev Quandles

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Introduction

Quandles are non associative algebraic structures which arise in several different areas:

- **knot** theory;
- set theoretical **QYBE**/braided vector spaces;
- Pointed **Hofp algebras**/Nichols algebras;

Quandles

A **quandle** $(X, \cdot)$ is an idempotent left-distributive left-quasigroup:

(1) $L_x : y \mapsto x \cdot y$, is bijective for every $x \in X$;

(2) $x \cdot (y \cdot z) \approx (x \cdot y) \cdot (x \cdot z)$

(3) $x \approx x \cdot x$.

i.e. $L_x \in Aut(X)$ which fixes x for every $x \in X$. So:

$$LMlt(X) = \langle L_x,\ x \in X \rangle \leq Aut(X).$$

and we have $L_{xy} = L_x^{-1} L_y L_x$ for every $x, y \in X$.

$$Dis(X) = \langle L_x {L_y}^{-1},\ x, y \in X \rangle$$

is called **displacement group of** X. A lot of quandle-theoretical properties of X can be translated into group-theoretical properties of $Dis(X)$ and of its subgroups.

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Examples

(1) Let X be a set. Then $P_X = (X, \cdot)$ with $x \cdot y = y$ for every $x, y \in X$, is called **projection quandle** over X.

(2) Let G be a group and $H \subset G$ a subset closed under **conjugation**. Then $Conj(H) = (H, \cdot)$, where $x \cdot y = x^{-1}yx$ is a quandle.

(3) In particular if X is a quandle and $L(X) = \{L_x,\ x \in X\} \subset LMlt(X)$ then $Conj(L(X))$ is a quandle.

(4) Let G be a group, $\alpha \in Aut(G)$ and $H \leq Fix(\alpha)$. Then $X = \mathcal{Q}(G, H, \alpha) = (G/H, \cdot)$ where

$$xH \cdot yH = x\alpha(x^{-1}y)H$$

is a quandle and it is said to be a **coset quandle**.

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Connected quandles

(1) Let X be a quandle. The set of orbits under the action of $LMlt(X)$ ($Dis(X)$) endowed with the structure of projection quandle is denoted by $\pi_0(X)$.

(2) A quandle X is **connected** if $LMlt(X)$ ($Dis(X)$) is transitive on X.

Lemma

Let X be a quandle. The map

$$\pi_0 : X \longrightarrow \pi_0(X) \quad x \mapsto x^{Dis(X)}$$

is a quandle morphism and $ker(\pi_0)$ is the smallest congruence of X with projection factor.

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Connected quandles

Corollary

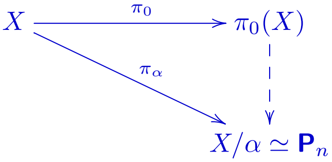
Let X be a quandle. Then the following are equivalent:

(1) X *is connected;*

(2) $|\pi_0(X)| = 1$;

(3) $\mathbf{P}_2 \notin \mathcal{H}(X)$.

Proof



Malt'sev term

Let X an idempotent algebra. A **Malt'sev** term, is an ternary term m, such that

$$m(x, y, y) \approx m(y, y, x) \approx x.$$

Theorem

[1] *Let X be an idempotent algebra. The following are equivalent:*

(1) *X is a Malt'sev algebra, i.e. X has a Malt'sev term.*

(2) *$\mathcal{V}(X)$ is congruence permutable.*

[1] Malt'sev (1954)

Taylor term

Let X be an idempotent algebra. A **Taylor** term, is an n-ary term t, such that for every coordinate $i \leq n$, t satisfies an identity of the form:

$$t(x_1, \dots, x_n) \approx t(y_1, \dots, y_n),$$

where $x_j, y_j \in \{x, y\}$ and $x_i = x$, $y_i = y$.

Theorem

[2] *Let X be an idempotent algebra. The following are equivalent:*

(1) X *is a Taylor algebra, i.e.* X *has a Taylor term.*

(2) $\mathcal{V}(X)$ *does not contain a two-element projection algebra.*

If X is finite, X has a Taylor term if and only if $\mathcal{HS}(X)$ does not contain a two-element projection algebra.

[2] Taylor (1977) - Barto, Kozik, Stanovský (2015)

Malt'sev quandles = Taylor quandles

Problem

Give a characterization of Taylor and Malt'sev quandles.

Lemma

Let X be a Taylor quandle. Then $\mathbf{P}_2 \notin \mathcal{HS}(X)$, i.e. all the subquandles of X are connected.

Corollary

Let X be a finite quandle. Then X is Taylor if and only if $\mathbf{P}_2 \notin \mathcal{HS}(X)$.

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Congruences and subgroups

Proposition

Let X be a quandle and α be its congruence. The map

$$\pi_\alpha^* : Dis(X) \longrightarrow Dis(X/\alpha), \quad L_x L_y^{-1} \mapsto L_{[x]} L_{[y]}^{-1}$$

is a surjective group morphism and $[h(x)]_\alpha = \pi_\alpha^*(h)([x]_\alpha)$ for every $h \in Dis(X)$ and every $x \in X$.

We define:

(1) $ker(\pi_\alpha^*) = K^\alpha(X)$;

(2) $D_\alpha(X) = \langle L_x L_y^{-1},\ x\ \alpha\ y \rangle \leq K^\alpha(X)$.

Note that:

(i) $D_\alpha(X)$ and $K^\alpha(X)$ are normal subgroups of $LMlt(X)$;

(ii) $x^{K^\alpha(X)} \subseteq [x]_\alpha$;

(iii) the restriction map $K^\alpha(X) \longrightarrow Aut([x]_\alpha)$, is a group morphism and $Dis([x]_\alpha) \leq D_\alpha(X)|_{[x]_\alpha}$ for every $x \in X$.

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Malt'sev quandles = Taylor quandles

Theorem

Let X be a Taylor quandle. Then it is a Malt'sev quandle.

Proof

Let α, β be congruences of X. Since X is Taylor, $D_\alpha(X)$ is transitive on every block of α, so $[x]_\alpha = x^{D_\alpha(X)}$ for every $x \in X$ (and the same for β). Hence,

$$[x]_{\alpha\circ\beta} = x^{D_\alpha(X)D_\beta(X)} = x^{D_\beta(X)D_\alpha(X)} = [x]_{\beta\circ\alpha}$$

Therefore, $\alpha \circ \beta = \beta \circ \alpha$. Since any $Y \in \mathcal{V}(X)$ is Taylor, then $\mathcal{V}(X)$ is congruence permutable.

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Meet-semidistributivity

Proposition

[3] *Let X be a finite idempotent algebra. Then the following are equivalent:*

(1) $\mathcal{V}(X)$ *is congruence meet-semidistributive;*

(2) $\mathcal{HS}(X)$ *does not contain any strictly simple Abelian algebra.*

Proposition

[4] *Let X be a finite quandle. The following are equivalent:*

(1) X *is simple Abelian;*

(2) X *is strictly simple.*

Theorem

No finite quandle is congruence meet-semidistributive.

[3] Larose, Valeriote

[4] (1) $\Rightarrow$ (2) Valeriote

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Summary and open problems

We know:

- For quandles **Taylor** $\Leftrightarrow$ **Malt'sev** $\Leftrightarrow$ congruence modularity ($\Leftrightarrow$ few subpowers, in the finite case);
- A finite quandle X is Malt'sev if and only if $\mathbf{P}_2 \notin \mathcal{HS}(X)$.
- No finite quandle is congruence **meet-semidistributive**.

We don't know:

- Give a general characterization of Malt'sev quandles ($\mathbf{P}_2 \notin \mathcal{HS}(X)$??).
- Do there exist (infinite) congruence meet-semidistributive quandles?

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