$\aleph_0$-categorical semigroups

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Semigroup basics

- Let S denote a semigroup i.e. a set with an associative binary operation.
- If S has an element 0 such that $0a = a0 = 0$ for all $a \in S$ then S is a **semigroup with zero**.
- We may adjoin a zero to S to form the semigroup with zero, denoted S^0.
- We say that $e \in S$ is an **idempotent** if $e = e^2$. We let $E(S)$ denote the set of all idempotents of S.
- We call S **regular** if for every $a \in S$ there exists $b \in S$ such that $a = aba$.
- We call S **orthodox** if it is regular and $E(S)$ forms a subsemigroup of S.

First order structure

Definition

A (first order) *structure* $\mathcal{M}$ is a set M, called the *universe*, together with:

1. a set of finitary relations on M, that is, subsets of M^n;
2. a set of finitary functions on M, that is, maps $M^n \to M$;
3. a set of constant elements.

- **Example** A semigroup $(S, \cdot)$ is a set S together with a binary function $\cdot$. Not every set with a binary function is a semigroup, e.g. $(\mathbb{Z}, -)$.
- **Example** A group $(G, \cdot, ^{-1}, 1)$ is a set G together with a binary function $\cdot$, a unary function $^{-1}$ and a constant 1.
- **Example** A graph (V, E) is a set V together with a binary relation E.

$\aleph_0$-categoricity

Definition

A structure is $\aleph_0$-categorical if it can be characterized, within the class of countable structures of the same 'type', by its first order properties up to isomorphism.

- (A. Grzegorczyk, 1968) The direct product of a finite number of $\aleph_0$-categorical structures is $\aleph_0$-categorical.
- (J. Rosenstein, 1973) Any abelian group of bounded order is $\aleph_0$-categorical.
- (A. Apps, 1982) Reduced the problem of classifying $\aleph_0$-categorical characteristically simple groups to the non-abelian p-group case.
- However, little is known in the case of semigroups.

The Ryll-Nardzewski Theorem

- The main tool in showing $\aleph_0$-categoricity is a theorem due independently to Engeler, Ryll-Nardzewski and Svenonius, but is commonly stated as the Ryll-Nardzewski Theorem (RNT).

- The automorphism group of a structure $\mathcal{M}$ has a natural action on M^n, where elements of Aut($\mathcal{M}$) act component-wise on the set M^n. That is, if $\phi \in \text{Aut}(\mathcal{M})$ and $(a_1, \ldots, a_n) \in M^n$ then define

$$(a_1, \ldots, a_n)\phi := (a_1\phi, \ldots, a_n\phi).$$

Theorem (The Ryll-Nardzewski Theorem)

A countable structure $\mathcal{M}$ is $\aleph_0$-categorical if and only if for each $n \geq 1$ Aut($\mathcal{M}$) has finitely many orbits on its action on M^n.

- It follows that all finite semigroups are $\aleph_0$-categorical.

Null semigroups

The countably infinite null semigroup N, with multiplication $xy = 0$ for all $x, y \in N$, is $\aleph_0$-categorical. For each $n \geq 1$, define a relation $\sharp_n$ on N^n by $\underline{a} = (a_1, \ldots, a_n) \, \sharp_n \, (b_1, \ldots, b_n) = \underline{b}$ if and only if

$$a_i = a_j \Leftrightarrow b_i = b_j \quad \text{and} \quad a_i = 0 \Leftrightarrow b_i = 0$$

So if $\underline{a} \, \sharp_n \, \underline{b}$ then the map $a_i \mapsto b_i$, $0 \mapsto 0$ is a well defined bijection. Since automorphisms of N are simply bijections of N which fix 0, it follows that $\sharp_n$-equivalent n-tuples lie in the same orbit of the action of $\text{Aut}(N)$ on N^n. The result follows by the RNT as

$$|N^n/\sharp_n| = B_{n+1} < \aleph_0,$$

where B_k is the kth Bells number, and is the number of ways of partitioning a set of size k.

The Substructure of $\aleph_0$-categorical semigroups

The RNT allows us to understand the structure of an arbitrary $\aleph_0$-categorical semigroup:

Corollary (TQG)

Let S be an $\aleph_0$-categorical semigroup with set of idempotents E. Then the following holds

1. *S is periodic;*
2. $\mathcal{D} = \mathcal{J}$;
3. *E is non-empty and $\langle E \rangle$ is $\aleph_0$-categorical;*
4. *Maximal subgroups of S are $\aleph_0$-categorical;*
5. *S has finitely many maximal subgroups, up to isomorphism;*
6. *The principal factors of S are $\aleph_0$-categorical;*
7. *S has finitely many principal factors, up to isomorphism.*

Completely 0-simple semigroups

- A semigroup S is **0-simple** if $S^2 \neq \{0\}$ and if $\{0\}$ and $S \setminus \{0\}$ are the only $\mathcal{J}$-classes.
- **Example** Each non-kernel principal factor is 0-simple or null.
- An idempotent e is **primitive** if

$$ef = fe = f \neq 0 \Rightarrow e = f$$

for all $f \in E(S)$.

- A semigroup is **completely 0-simple** if it is 0-simple and contains a primitive idempotent.
- Every **periodic** 0-simple semigroup is completely 0-simple.

The Rees Theorem

Theorem (The Rees Theorem)

Let G^0 be a group with zero adjoined, let I,Λ be non-empty index sets and let $P = (p_{\lambda,i})$ be a $\Lambda \times I$ matrix with entries in G^0. Suppose no row or column of P consists entirely of zeros. Let $S = (I \times G \times \Lambda) \cup \{0\}$, and define multiplication on S by

$$(i, g, \lambda)(j, h, \mu) = \begin{cases} (i, g\, p_{\lambda,j}\, h, \mu) & \text{if } p_{\lambda,j} \neq 0, \\ 0 & \text{if } p_{\lambda,j} = 0. \end{cases}$$

$$0(i, g, \lambda) = (i, g, \lambda)0 = 00 = 0.$$

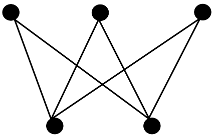
Then S is a completely 0-simple semigroup, and is called a Rees matrix semigroup, denoted $\mathcal{M}^0[G; I, \Lambda; P]$. Conversely, every completely 0-simple semigroup is isomorphic to a Rees matrix semigroup.

Bipartite graphs

Definition

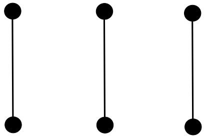
A *bipartite graph* is a graph (V, E) whose vertices can be split into two disjoint non-empty sets L and R such that every edge connects a vertex in L to a vertex in R. Formally, we consider Γ as the structure $(V; E, L, R)$ with a binary relation E and pair of unary relations L and R.

- **Example** A bipartite graph is *complete* if every vertex in L is adjacent to every vertex in R.



Bipartite graphs

- **Example** A bipartite graph in which each vertex is incident to exactly one edge is called a *perfect matching*.



- Homogeneous bipartite graphs have been classified by Goldstern (1996). For relational structures homogeneity is a stronger property than $\aleph_0$-categoricity, and examples include all complete bipartite graphs and perfect matchings.

Bipartite graph of Rees matrix semigroup

Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be a Rees matrix semigroup with sandwich matrix $P = (p_{\lambda,i})$. Then we may form a bipartite graph, denoted $\Gamma(P)$ with

1. set of vertices $I \cup \Lambda$,
2. an edge $(i, \lambda) \in I \times \Lambda$ whenever $p_{\lambda,i} \neq 0$.

This construct was first used by Graham (1968) to describe the maximal subsemigroups of a finite Rees matrix semigroup.

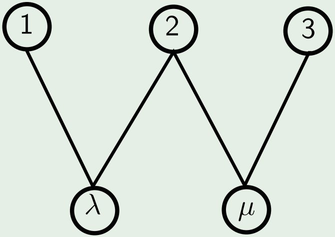
Bipartite graph of Rees matrix semigroup

Example

Let $S = \mathcal{M}^0[G; \{1,2,3\}, \{\lambda, \mu\}; P]$ where

$$P = \begin{pmatrix} a & b & 0 \\ 0 & c & d \end{pmatrix}.$$

Then $\Gamma(P)$ is



Automorphisms

Theorem

Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ *be a Rees matrix semigroup where* $P = (p_{\lambda,i})$*. Let* $\psi \in Aut(\Gamma(P))$ *and* $\theta \in Aut(G)$ *and* $u_i, v_\lambda \in G$ *for each* $i \in I, \lambda \in \Lambda$*. Then the mapping* $\phi : S \to S$ *given by*

$$(i, g, \lambda)\phi = (i\psi, u_i \cdot (g\theta) \cdot v_\lambda, \lambda\psi)$$

is an automorphism of S *if and only if*

$$p_{\lambda,i}\,\theta = v_\lambda \cdot p_{\lambda\psi,i\psi} \cdot u_i, \text{ whenever } p_{\lambda,i} \neq 0.$$

Moreover, every automorphism of S *can be described in this way.*

Corollary (TQG)

Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ *be an* $\aleph_0$*-categorical Rees matrix semigroup. Then* G *and* $\Gamma(P)$ *are* $\aleph_0$*-categorical.*

Pure Rees matrix semigroups

Definition

Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be a Rees matrix semigroup such that the matrix P is over $\{0, 1\}$, where 1 is the identity of G. Then S is called a **pure** Rees matrix semigroup. A semigroup isomorphic to a pure Rees matrix semigroups is called a **pure** completely 0-simple semigroup.

Theorem (C.H. Houghton)

A completely 0-simple semigroup S is pure if and only if, for all $a, b \in S$

$$[a, b \in \langle E \rangle \text{ and } a \,\mathcal{H}\, b] \Rightarrow a = b.$$

Theorem (TQG)

A pure Rees matrix semigroup $\mathcal{M}^0[G; I, \Lambda; P]$ is $\aleph_0$-categorical if and only if both G and $\Gamma(P)$ are $\aleph_0$-categorical.

$\aleph_0$-categorical orthodox Rees matrix semigroups

- Every orthodox completely 0-simple semigroup is pure, however the converse does not hold.
- (Hall, 1969) A completely 0-simple semigroup is orthodox if and only if it is isomorphic to a Rees matrix semigroup with sandwich matrix over $\{0, 1\}$ and with induced bipartite graph a disjoint union of complete bipartite graphs.

Corollary

Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be an orthodox Rees matrix semigroup, with P over $\{0, 1\}$. Let $\Gamma(P) = \bigsqcup_{i \in A} \Gamma_i$ with each Γ_i complete. Then S is $\aleph_0$-categorical if and only if G is $\aleph_0$-categorical and $\{\Gamma_i : i \in A\}$ is finite, up to isomorphism.

Hierarchy of results

Given a sandwich matrix P of a Rees matrix semigroup, we call an entry $p_{\lambda,i}$ of P **non-trivial** if $p_{\lambda,i} \notin \{0, 1\}$.

Condition on $\mathcal{M}^0[G; I, \Lambda; P]$	N & S for $\aleph_0$-categoricity
Inverse	G
Orthodox	G and $\Gamma(P)$
Pure	G and $\Gamma(P)$
P has finitely many non-trivial entries	G and $\Gamma(P)$
I or Λ finite	G

Table: The $\aleph_0$-categoricity of certain classes of Rees matrix semigroups

Insufficiency for the general result

- **Question:** Is G and $\Gamma(P)$ being $\aleph_0$-categorical sufficient for a Rees matrix semigroup to be $\aleph_0$-categorical? **No!**
- If $\langle E \rangle$ $\aleph_0$-categorical then so is $\Gamma(P)$. The converse does not hold in general.
- **Open problem:** Find a non-$\aleph_0$-categorical Rees matrix semigroup such that G and $\langle E \rangle$ are both $\aleph_0$-categorical.