Finite group presentations, van Kampen diagrams, and a combinatorial problem

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Notations

G - a finite group
$G = \langle S|R \rangle$ - a presentation of the group G, with generating set S, and relators set R
$\mathbb{N}$ - the set of nonnegative integers
$\mathcal{D}(G)$ - the set of distribution functions on the group G
$[a]_\sim$ - the equivalence class of the element a with respect to the equivalence relation $\sim$

Group presentations

Remark

A presentation $G = \langle S|R\rangle = \langle s_1, s_2, \ldots, s_k, \ldots | r_1(s_1, s_2, \ldots, s_k, \ldots), \ldots, r_1(s_1, s_2, \ldots, s_k, \ldots), \ldots\rangle$ of a group G describes the fact that $G \cong F/N$, where F is a free group over a set X of cardinal $|S|$, and N is the normal closure of the set of words $R(X) = \{r_1(x_1, \ldots, x_l, \ldots), \ldots, r_l(x_1, \ldots, x_l, \ldots), \ldots\}$. If both S and R are finite, this is called a finite presentation of G. Any finite group is finitely presented.

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$R(X) = \{r_1(x_1, \ldots, x_l, \ldots), \ldots, r_l(x_1, \ldots, x_l, \ldots), \ldots\}$. If both S and R are finite, this is called a finite presentation of G. Any finite group is finitely presented.

van Kampen diagram associated with a presentation of a group

Definition

Let
$G = \langle s_1, s_2, \ldots, s_k | r_1(s_1, \ldots, s_k), r_2(s_1, \ldots, s_k), \ldots, r_l(s_1, \ldots, s_k) \rangle$
be a finite presentation of a group. The van Kampen diagram of G associated with this presentation is a directed graph with vertex set $V = G$ and edge set $E = \{(x, xs_i) | x \in G, i = \overline{1, k}\}$.

Distribution functions

Definition

Let G be a finite group. Any function $f : G \longrightarrow \mathbb{N}$ will be a called a distribution function on G. We shall denote $\mathcal{D}(G)$ the set of distribution functions on G. If $f \in \mathcal{D}(G)$ and $X \subseteq G$, the weight of X with respect to f is

$$w_f(X) = \sum_{x \in X} f(x).$$

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Redistributions

Definition

Let G be a finite group, with a finite presentation $G = \langle s_1, \ldots, s_k | r_1, \ldots, r_l \rangle$, and $f, g \in \mathcal{D}(G)$. We say that f can be redistributed into g if there is a function $\rho : \bigcup_{i=1}^{k} \{(x, xs_i), (x, xs_i^{-1}) | x \in G\} \longrightarrow \mathbb{N}$, called redistribution function, such that for each $x \in G$ the following equalities hold:

$$f(x) = \sum_{i=1}^{k} \rho(x, xs_i) + \sum_{i=1}^{k} \rho(x, xs_i^{-1}),$$

$$g(x) = \sum_{i=1}^{k} \rho(xs_i^{-1}, x) + \sum_{i=1}^{k} \rho(xs_i, x).$$

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Associated distributions

Definition

Let $G = \langle s_1, \ldots, s_k | r_1, \ldots, r_l \rangle$ be a presentation of a finite group G, and $f, g \in \mathcal{D}(G)$. We say that f and g are associated with respect to the given presentation if $f = g$ or there is a finite sequence $h_0, h_1, \ldots, h_n$ of distribution functions and a finite sequence $\rho_1, \ldots, \rho_n$ of redistribution functions, such that $h_0 = f$, $h_n = g$, and $h_i = h_{i-1} \cdot \rho_i$, $\forall i = \overline{1, n}$. We write $f \sim g$ in such a case.

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Problem

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Given a finite group G, presented as $G = \langle s_1, \ldots, s_k | r_1, \ldots, r_l \rangle$, and a distribution function $f \in \mathcal{D}(G)$, determine the equivalence class $[f]_\sim$ and its cardinal $|[f]_\sim|$.

Remark

If $G = \langle s_1, \ldots, s_k | r_1, \ldots, r_l \rangle$ is a finitely presented group, and $f, g \in \mathcal{D}(G)$, such that $f \sim g$, then $w(f) = w(g)$.

A lemma

Proposition

Let $G = \langle s_1, \ldots, s_k | r_1, \ldots, r_l \rangle$ *be a finitely presented group,* $f, g \in \mathcal{D}(G)$*, and* $x, y \in G$*, such that there is a path of length* 2 *between* x *and* y *in the nondirected graph underlying the van Kampen diagram of the presentation. If* $w(f) = w(g)$*,* f *and* g *coincide on* $G \setminus \{x, y\}$*, and* $f(x) - g(x) = 1$*, then* $f \sim g$*.*

Results from nondirected graphs

Proposition

Let $G = (V = A \cup B, E)$ be a finite connected, nondirected, bipartite graph, and $f, g \in \mathcal{D}(G)$. Then $f \sim g$ if and only if $w_f(A) = w_g(A)$ and $w_f(B) = w_g(B)$, or $w_f(A) = w_g(B)$ and $w_f(B) = w_g(A)$.

Proposition

Let $G = (V, E)$ be a finite connected nondirected graph, which is not bipartite, and $f, g \in \mathcal{D}(G)$. Then $f \sim g$ if and only if $w(f) = w(g)$.

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Proposition

Let G be a finite group, which is not a $2-$group. Then the nondirected graph underlying its van Kampen diagram, associated to an arbitrary presentation, cannot be a bipartite graph.

Corollary

Let G be a finite group, which is not a $2-$group, and $f, g \in \mathcal{D}(G)$. Then $f \sim g$ if and only if $w(f) = w(g)$.

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$$|[f]_{\sim}| = \binom{w(f) + |G| - 1}{w(f)}.$$

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The case of $2-$groups

Proposition

Let $G = \langle s_1, \ldots, s_k | r_1, \ldots, r_l \rangle$ be a finitely presented $2-$group. If the presentation of G is minimal on the number of generators, then there are no cycles of odd length in the van Kampen diagram associated with the presentation.

Corollary

Let $G = \langle s_1, \ldots, s_k | r_1, \ldots, r_l \rangle$ be a finitely presented $2-$group, such that the presentation is minimal on the number of generators. Then the van Kampen diagram associated with the presentation is a bipartite graph. The partition of G is given by $A = \{x \in G | x \text{ is a product of an even number of generators}\}$, and $B = G \setminus A$.

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Corollary

In the above conditions, if $a = w_f(A)$ and $b = w_f(B)$, then

$$|[f]_\sim| = \binom{a + |A| - 1}{a} \cdot \binom{b + |B| - 1}{b} +$$

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Thank you!