# CANCELLABLE ELEMENTS OF LATTICES OF SEMIGROUP AND EPIGROUP VARIETIES

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# Neutral elements

An element $x$ of a lattice $L$ is called *neutral* if

$$\forall y, z \in L: \quad \text{the sublattice of all generated by } x, y \text{ and } z \text{ is distributive}$$

or, equivalently, if

$$\forall y, z \in L: \quad (x \vee y) \wedge (y \vee z) \wedge (z \vee x) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x).$$

If $a$ is neutral in $L$ then $L$ is a subdirect product of $(a]$ and $[a)$,

$L$ embeds in $(a] \times [a)$ by the rule

$$x \longmapsto (x \wedge a, x \vee a) \text{ for any } x \in L.$$

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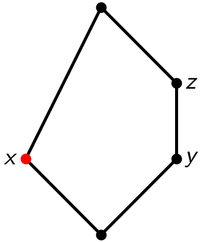
An element $x$ of a lattice $L$ is called *modular* if

$$\forall y, z\colon \quad y \leq z \to (x \vee y) \wedge z = (x \wedge z) \vee y.$$




This configuration is impossible

An element $x$ of a lattice $L$ is called *cancellable* if

$$\forall y, z: \quad x \wedge y = x \wedge z \;\&\; x \vee y = x \vee z \rightarrow y = z.$$

Every neutral element is cancellable.

Every cancellable element is neutral.

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Let **SEM** be the lattice of all semigroup varieties.

**Proposition 1 (Jezek and McKenzie, 1993; reproved in simpler way by Shaprynskiĭ, 2012)**

*If* **V** *is a modular element of the lattice* **SEM** *then either* **V** *is the variety of all semigroups or* $\mathbf{V} \subseteq \mathbf{SL} \vee \mathbf{N}$ *where* **SL** *is the variety of semilattices, while* **N** *is a nilvariety.*

**Proposition 2 (∼, 2007)**

*A commutative semigroup variety* **V** *is a modular element of the lattice* **SEM** *if and only if* $\mathbf{V} \subseteq \mathbf{SL} \vee \mathbf{N}$ *where* **N** *satisfies the identities* $x^2y = 0$ *and* $xy = yx$.

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# Cancellable elements in the lattice **SEM**: the commutative case

**Theorem**

*For a commutative semigroup variety* **V**, *the following are equivalent:*

a) **V** *is a cancellable element of the lattice* **SEM**;
b) **V** *is a modular element of the lattice* **SEM**;
c) $\mathbf{V} \subseteq \mathbf{SL} \vee \mathbf{N}$ *where* **N** *satisfies the identities* $x^2y = 0$ *and* $xy = yx$.

($w = 0$ means $wx = xw = w$ where $x$ does not occur in $w$)

**Question 1**

Does there exist a semigroup variety that is a modular but not cancellable element of the lattice **SEM**?

Proposition 1 completely reduces the problem of description of modular elements in **SEM** to the nil-case.

An important class of nilvarieties: a variety is called *0-reduced* if it is given by identities of the form $w = 0$.

**Proposition 3(∼ and Volkov, 1988; independently, Jezek and McKenzie, 1993)**

*A 0-reduced semigroup variety is a modular element of the lattice* **SEM**.

**Question 2**

Is a 0-reduced semigroup variety a cancellable element of the lattice **SEM**?

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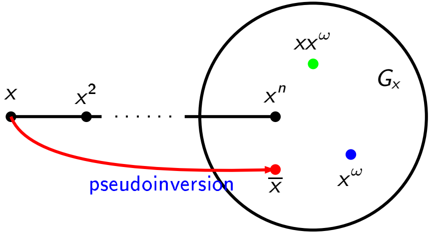
Let $S$ be an epigroup, $x \in S$, $G_x$ is the maximum subgroup of $S$ containing $x$. Let $x^\omega$ be a unit element of $G_x$. Then $xx^\omega = x^\omega x \in G_x$. Put

$$\overline{x} = (xx^\omega)^{-1} \text{ in } G_x.$$

$\overline{x}$ is called *pseudoinverse* to $x$



Every periodic semigroup variety can be considered as a variety of epigroups.

If an epigroup variety **V** consists of periodic semigroups then the operation of pseudoinversion may be defined by multiplication. Namely, if **V** satisfies the identity $x^m = x^{m+n}$ then $\overline{x} = x^{(m+1)n-1}$. Thus a variety of periodic epigroups can be considered as epigroup variety.

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Let **EPI** be the lattice of all epigroup varieties.

**Theorem**

*For a commutative epigroup variety* **V**, *the following are equivalent:*

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The equivalence of b) and c) was proved earlier by $\sim$, Skokov and Shaprynskii (2016).

**Corollary**

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