Cut-continuous pomonoids

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This is joint work with D. Kruml and J. Paseka (Masaryk University, Brno).
A **residuated lattice** is an algebra \((L; \wedge, \lor, \cdot, /, \backslash, 1)\) such that

1. \((L; \wedge, \lor)\) is a lattice;
2. \((L; \cdot, 1)\) is a monoid;
3. left, right multiplication is residuated:
   for any \(x, y\), \(\{c : y \cdot c \leq x\}\) and \(\{c : c \cdot y \leq x\}\) are principal ideals,
   generated by \(y \backslash x\) and \(x / y\), respectively.
Definition

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A residuated lattice is called

- commutative if so is \(\cdot\),
- integral if 1 is the top element.

In what follows, commutativity and integrality will be understood.
Definition

A **filter** of a residuated lattice is an upwards closed subalgebra.
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Proposition (McCarthy; Blount, Tsinakis)
Let $F$ be a filter of a residuated lattice $L$. Define, for $a, b \in L$,
\[
    a \theta_F b \quad \text{if } a f \leq b \text{ and } b f \leq a \text{ for some } f \in F.
\]
Then $\theta_F$ is a congruence and $F = 1/\theta_F$.
All congruences on $L$ arise in this way.
Definition

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Challenge
Given residuated lattices $P$ and $F$, determine the coextensions of $P$ by $F$. 
Let $L$ be a residuated lattice and $F$ a filter of $L$.

The product on $L$ splits into the following mappings:

1. $\cdot : F \times F \rightarrow F$.
2. $\cdot : F \times R \rightarrow R$, where $R$ is a congruence class $\neq F$.
3. $\cdot : R \times S \rightarrow R \cdot S$, where $R, S$ are congruence classes $\neq F$. 
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Ad (1) $\cdot : F \times F \to F$

This is the product of $F$. 
Definition

Let $F$ be a residuated lattice. An $F$-module is a $\vee$-semilattice $R$ together with a mapping $\star : F \times R \to R$ such that

- $\star$ is residuated in each argument,
- $f \star (g \star r) = f \cdot g \star r$ for any $r \in R$ and $f, g \in F$ and $1 \star r = r$ for any $r \in R$. 

Notes.
- This is the "residuated" version of an $S$-poset (Fakhruddin).
- Replacing $F$ by a quantale $Q$ and "residuated" by "join-preserving", this is a $Q$-module (Abramsky, Vickers).

Ad (2)· $F \times R \to R$, where $R$ is a congruence class $\neq F$. This makes $R$ into an $F$-module: $f \star r = f \cdot r$, $f \in F$, $r \in R$. 


Definition

Let \( F \) be a residuated lattice.

An \( F \)-module is a \( \lor \)-semilattice \( R \) together with a mapping \( \star : F \times R \to R \) such that

- \( \star \) is residuated in each argument,
- \( f \star (g \star r) = fg \star r \) for any \( r \in R \) and \( f, g \in F \)
  and \( 1 \star r = r \) for any \( r \in R \).

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Ad (2) $\cdot : F \times R \to R$, where $R$ is a congruence class $\not= F$. This makes $R$ into an $F$-module: $f \star r = f \cdot r$, $f \in F$, $r \in R$. 
Homomorphisms of $F$-modules

**Definition**

Let $R$ and $S$ be $F$-modules. Then $\varphi : R \to S$ is a **homomorphism** if $\varphi$ is residuated and

$$\varphi(f \triangleright r) = f \triangleright \varphi(r)$$

for any $f \in F$ and $r \in R$. 
Bihomorphisms of $F$-modules

**Definition**

Let $R$, $S$, and $T$ be $F$-modules. Then $\psi: R \times S \to T$ is a bihomomorphism if $\psi$ is a homomorphism in each argument.
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Let $R$, $S$, and $T$ be $F$-modules. Then $\psi: R \times S \to T$ is a bihomomorphism if $\psi$ is a homomorphism in each argument.

Still, let $L$ be a residuated lattice and $F$ a filter of $L$.

**Ad (3):** $\cdot : R \times S \to R \cdot S$, where $R, S$ are congruence classes $\neq F$. If $S < R\setminus(R \cdot S)$ or $R < (R \cdot S)/S$, this mapping is trivial. Otherwise, $R$ and $S$ being viewed as $F$-modules, this mapping is a bihomomorphism from $R \times S$ to $R \cdot S$. 
Assume we are given residuated lattices $P$ and $F$.

In order to determine a coextension of $P$ by $F$, we need:

- for each element of $r \in P$, an $F$-module $M_r$;
- for each (relevant) $r, s, t \in P$ such that $r \cdot s = t$, a bihomomorphism $M_r \times M_s \to M_t$. 
We consider the particular case of $F = \mathbb{R}^\times$. 
We consider the particular case of $F = \mathbb{R}^-$. 

**Proposition (M. Broušek, Th. V.)**

Let $M$ be a totally ordered, non-trivial $\mathbb{R}^-$-module. Then $M$ is order-isomorphic to one of $\mathbb{R}^-$, $\mathbb{R}$, $\mathbb{R}^+$, or $[u,0]$, where $u < 0$, and under this isomorphism, the action is the (truncated) addition of reals.
Definition

A tensor product of $F$-modules $R$ and $S$ is an $F$-module $R \otimes_F S$ together with a bihomomorphism $\pi: R \times S \rightarrow R \otimes_F S$ such that:

$$R \times S \xrightarrow{\pi} R \otimes_F S$$

$$f \downarrow \quad \tilde{f}$$

For any bihomomorphism $f : R \times S \rightarrow T$, there is a unique homomorphism $\tilde{f} : R \otimes_F S \rightarrow T$ such that $f = \tilde{f} \circ \pi$. 
Negative results

The tensor product of modules over residuated lattices does in general not exist.

Proposition (E. Nelson)
In the category of bounded posets and residuated mappings, a tensor product does not exist.
A generalised setting

Let $P$ be a poset.

For $A \subseteq P$, let $A^{\uparrow\downarrow}$ be the cut generated by $A$, that is, the set of lower bounds of its upper bounds.
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The mapping

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is a closure operator on $P$, making $P$ into a closure space.
Let $P$ be a poset.

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**Definition (A.A. Bishop, M. Erné)**

A map $f: P \to Q$ between posets $P$ and $Q$ is called **cut-continuous** if $f$ is a continuous map between the closure spaces $P$ and $Q$:

$$f(A^{↑↓}) \subseteq f(A)^{↑↓} \quad \text{for any } A \subseteq P.$$
Lemma

Let $f : P \to Q$ be a map between posets.

$f$ is residuated iff
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Let $f : P \to Q$ be a map between posets.

$f$ is residuated iff
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$f$ is cut-continuous iff
the inverse image of any principal ideal is a cut.
**Definition**

A **cut-continuous pomonoid** is an algebra \((L; \land, \lor, \cdot, 1)\) such that:

1. \((L; \land, \lor)\) is a lattice;
2. \((L; \cdot, 1)\) is a monoid;
3. left, right multiplication is **cut-continuous**: for any \(x, y\), the sets \(\{c: y \cdot c \leq x\}\) and \(\{c: c \cdot y \leq x\}\) are cuts.

A cut-continuous pomonoid is called

- **commutative** if so is \(\cdot\),
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In what follows, commutativity and integrality will be understood.
**Definition**

A **filter** of a cut-continuous pomonoid is an upwards closed subalgebra.

**Proposition (D. Kruml, J. Paseka, Th.V.)**

Let $\theta$ be a congruence on a cut-continuous pomonoid. Then $F = 1/\theta$ is a filter and

$$a \theta_F b \quad \text{if} \quad af \leq b \text{ and } bf \leq a \text{ for some } f \in F.$$
Coextensions

Definition
Let $L$ be a cut-continuous pomonoid and let $P$ be the quotient of $L$ by a filter $F$. Then we call $L$ the coextension of $P$ by $F$.

Challenge
Given cut-continuous pomonoids $P$ and $F$, determine the coextensions of $P$ by $F$. 
Theorem (M. Erné, J. Picado)

Closure spaces $A$ and $B$ possess a **tensor product** – a map from $A \times B$ to a closure space $A \otimes B$ such that:

$$A \times B \xrightarrow{\pi} A \otimes B$$

For any separately continuous map $f$ from $A \times B$ to a sup-lattice $C$, there is a unique join-preserving map $\tilde{f}: A \otimes B \to C$ such that $f = \tilde{f} \circ \pi$. 
Theorem (D. Kruml, J. Paseka, Th. V.)

Let $F$ be a cut-continuous pomonoid. $F$-modules $R$ and $S$ possess a tensor product – a bihomomorphism $\pi$ from $R \times S$ to $R \otimes_F S$ such that:

$$
\begin{array}{ccc}
R \times S & \xrightarrow{\pi} & R \otimes_F S \\
\downarrow f & & \downarrow \tilde{f} \\
T & & 
\end{array}
$$

For any bimorphism $f$ from $R \times S$ to an $F$-module and sup-lattice $T$, there is a morphism $\tilde{f}: R \otimes_C S \to T$ such that $f = \tilde{f} \circ \pi$. 
Consider the cut-continuous pomonoid \((\mathbb{R}^{-}; \leq, +, 0)\).
Then \(\mathbb{R}^{-}\) becomes a \(\mathbb{R}^{-}\)-module by usual addition of reals.
Simple example of a tensor product

Consider the cut-continuous pomonoid \((\mathbb{R}^{-}; \leq, +, 0)\).
Then \(\mathbb{R}^{-}\) becomes a \(\mathbb{R}^{-}\)-module by usual addition of reals.
The tensor product of \(\mathbb{R}^{-}\) with itself is

\[\mathbb{R}^{-} \otimes_{\mathbb{R}^{-}} \mathbb{R}^{-} = \mathbb{R}^{-},\]

with

\[\pi: \mathbb{R}^{-} \times \mathbb{R}^{-} \rightarrow \mathbb{R}^{-}, \ (r, s) \mapsto r + s.\]
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\[ K = \{0, 1, 2, 3, 4\} \]

by \( \mathbb{R}^- \), such that the result is order-isomorphic to \([0, 1]\).
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Our procedure:

- We choose the order-type of the \( \mathbb{R}^- \)-modules, e.g.:

  \[ \mathbb{R}^+, \{0\}, \mathbb{R}^-, \mathbb{R}^-, \mathbb{R}^- \].
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Our procedure:

- We choose the order-type of the \( \mathbb{R}^- \)-modules, e.g.:
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- We choose the parameters of the homomorphisms between the occurring \( \mathbb{R}^- \)-modules.
The result is a **triangular norm**:

\[
    a \odot b =
\begin{cases}
    a \wedge b & \text{if } a, b > \frac{3}{4}, \\
    a \wedge (b - \frac{1}{4}) & \text{if } b > \frac{3}{4} \text{ and } \frac{1}{2} < a \leq \frac{3}{4}, \\
    a \wedge (b - \frac{1}{2}) & \text{if } b > \frac{3}{4} \text{ and } \frac{1}{4} < a \leq \frac{1}{2}, \\
    a & \text{if } b > \frac{3}{4}, a \leq \frac{1}{4}, \text{ and } a + b > 1, \\
    0 & \text{if } b \leq \frac{3}{4}, a \leq \frac{1}{4}, \text{ and } a + b \leq 1, \\
    (a - \frac{1}{4}) \wedge (b - \frac{1}{4}) \wedge \frac{7}{16} & \text{if } \frac{1}{2} < a, b \leq \frac{3}{4}, \\
    \frac{1}{8} & \text{if } \frac{5}{8} < b \leq \frac{3}{4} \text{ and } \frac{3}{8} < a \leq \frac{1}{2}, \\
    0 & \text{or } b \leq \frac{5}{8} \text{ and } a \leq \frac{1}{2}.
\end{cases}
\]
Summary

- Cut-continuous pomonoids generalise residuated lattices.
- A tensor product of modules over cut-continuous pomonoids exists.
- The construction of coextensions is in this way facilitated.

Problems and ongoing work

The universality of the tensor product is restricted to mappings to sup-lattices.

The plan is elaborate on the case of conditionally complete lattices.

The shown application to coextensions restricts to the totally ordered case.
Conclusion

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