Cut-continuous pomonoids

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This is joint work with D. Kruml and J. Paseka
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Motivation

Definition

A residuated lattice is an algebra $(L; \wedge, \vee, \cdot, /, \backslash, 1)$ such that

(1) $(L; \wedge, \vee)$ is a lattice;

(2) $(L; \cdot, 1)$ is a monoid;

(3) left, right multiplication is residuated:
for any x, y, $\{c\colon y \cdot c \leqslant x\}$ and $\{c\colon c \cdot y \leqslant x\}$ are principal ideals,
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A residuated lattice is called

commutative if so is $\cdot$,

integral if 1 is the top element.

In what follows, commutativity and integrality will be understood.

Quotients

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Proposition (McCarthy; Blount, Tsinakis)

Let F be a filter of a residuated lattice L. Define, for $a, b \in L$,

$$a \, \theta_F \, b \quad \text{if } a\,f \leqslant b \text{ and } b\,f \leqslant a \text{ for some } f \in F.$$

Then θ_F is a congruence and $F = 1/\theta_F$.

All congruences on L arise in this way.

Coextensions

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Then we call L a coextension of P by F.

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Challenge

**Given residuated lattices P and F,
determine the coextensions of P by F.**

“Local” description of a coextension

Let L be a residuated lattice and F a filter of L.

The product on L splits into the following mappings:

(1) $\cdot \colon F \times F \to F$.

(2) $\cdot \colon F \times R \to R$, where R is a congruence class $\neq F$.

(3) $\cdot \colon R \times S \to R \cdot S$, where R, S are congruence classes $\neq F$.

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Ad (1) $\cdot\colon F \times F \to F$

This is the product of F.

Modules over a residuated lattice

Definition

Let F be a residuated lattice.
An F-module is a $\vee$-semilattice R
together with a mapping $\star\colon F \times R \to R$ such that

- $\star$ is residuated in each argument,
- $f \star (g \star r) = f\, g \star r$ for any $r \in R$ and $f, g \in F$ and $1 \star r = r$ for any $r \in R$.

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Notes.

- This is the "residuated" version of an *S-poset* (Fakhruddin).
- Replacing F by a quantale Q and "residuated" by "join-preserving", this is a *Q-module* (Abramsky, Vickers).

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Ad (2) $\cdot \colon F \times R \to R$, where R is a congruence class $\neq F$.

This makes R into an F-module: $f \star r \;=\; f \cdot r, \quad f \in F,\ r \in R.$

Homomorphisms of F-modules

Definition

Let R and S be F-modules.
Then $\varphi\colon R \to S$ is a homomorphism if φ is residuated and

$$\varphi(f \star r) \;=\; f \star \varphi(r)$$

for any $f \in F$ and $r \in R$.

Bihomorphisms of F-modules

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Let R, S, and T be F-modules.
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Still, let L be a residuated lattice and F a filter of L.

Ad (3): $\cdot \colon R \times S \to R \cdot S$, where R, S are congruence classes $\neq F$.

If $S < R \backslash (R \cdot S)$ or $R < (R \cdot S)/S$, this mapping is trivial.

Otherwise, R and S being viewed as F-modules,
this mapping is a bihomomorphism from $R \times S$ to $R \cdot S$.

Construction of a coextension

Assume we are given residuated lattices P and F.

In order to determine a coextension of P by F, we need:

- for each element of $r \in P$, an F-module M_r;
- for each (relevant) $r, s, t \in P$ such that $r \cdot s = t$, a bihomomorphism $M_r \times M_s \to M_t$.

Construction of F-modules

We consider the particular case of $F = \mathbb{R}^-$.

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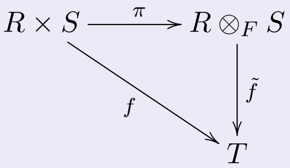
Proposition (M. Broušek, Th. V.)

Let M be a totally ordered, non-trivial $\mathbb{R}^-$-module. Then M is order-isomorphic to one of $\mathbb{R}^-$, $\mathbb{R}$, $\mathbb{R}^+$, or $[u, 0]$, where $u < 0$, and under this isomorphism, the action is the (truncated) addition of reals.

Construction of bihomomorphisms

Definition

A tensor product of F-modules R and S is an F-module $R \otimes_F S$ together with a bihomomorphism $\pi \colon R \times S \to R \otimes_F S$ such that:



For any bihomomorphism $f \colon R \times S \to T$, there is a unique homomorphism $\tilde{f} \colon R \otimes_F S \to T$ such that $f = \tilde{f} \circ \pi$.

Negative results

The tensor product of modules over residuated lattices does in general not exist.

Proposition (E. Nelson)

In the category of bounded posets and residuated mappings, a tensor product does not exist.

A generalised setting

Let P be a poset.

For $A \subseteq P$, let $A^{\uparrow\downarrow}$ be the cut generated by A, that is, the set of lower bounds of its upper bounds.

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Definition (A.A. Bishop, M. Erné)

A map $f\colon P \to Q$ between posets P and Q is called cut-continuous if f is a continuous map between the closure spaces P and Q:

$$f(A^{\uparrow\downarrow}) \subseteq f(A)^{\uparrow\downarrow} \quad \text{for any } A \subseteq P.$$

Residuated vs. cut-continuous

Lemma

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Let $f\colon P \to Q$ be a map between posets.

f is residuated iff
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f is cut-continuous iff
the inverse image of any principal ideal is a cut.

Cut-continuous pomonoids

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A cut-continous pomonoid is an algebra $(L; \wedge, \vee, \cdot, 1)$ such that:

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Then $F = 1/\theta$ is a filter and

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Given cut-continuous pomonoids P and F,
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Tensor product of closure spaces

Theorem (M. Erné, J. Picado)

Closure spaces A and B possess a tensor product –
a map from $A \times B$ to a closure space $A \otimes B$ such that:

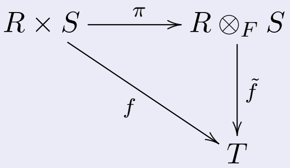
$$\begin{array}{ccc} A \times B & \xrightarrow{\ \pi\ } & A \otimes B \\ & \searrow^{f} & \downarrow{\scriptstyle \tilde{f}} \\ & & C \end{array}$$

For any separately continuous map f from $A \times B$ to a sup-lattice C, there is a unique join-preserving map $\tilde{f} \colon A \otimes B \to C$ such that $f = \tilde{f} \circ \pi$.

Tensor product of modules over cut-continuous pomonoids

Theorem (D. Kruml, J. Paseka, Th. V.)

Let F be a cut-continuous pomonoid.
F-modules R and S possess a tensor product –
a bihomomorphism π from $R \times S$ to $R \otimes_F S$ such that:



For any bimorphism f from $R \times S$ to an F-module and sup-lattice T, there is a morphism $\tilde{f}\colon R \otimes_C S \to T$ such that $f = \tilde{f} \circ \pi$.

Simple example of a tensor product

Consider the cut-continuous pomonoid $(\mathbb{R}^-; \leqslant, +, 0)$.

Then $\mathbb{R}^-$ becomes a $\mathbb{R}^-$-module by usual addition of reals.

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Then $\mathbb{R}^-$ becomes a $\mathbb{R}^-$-module by usual addition of reals.

The tensor product of $\mathbb{R}^-$ with itself is

$$\mathbb{R}^- \otimes_{\mathbb{R}^-} \mathbb{R}^- \; = \; \mathbb{R}^-,$$

with

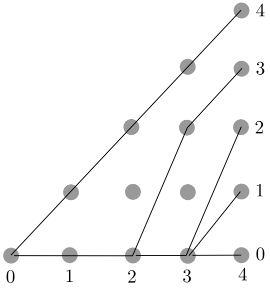
$$\pi\colon \mathbb{R}^- \times \mathbb{R}^- \to \mathbb{R}^-, \; (r, s) \mapsto r + s.$$

Example of a coextension

We want to determine the coextensions of

$$K = \{0, 1, 2, 3, 4\}$$

by $\mathbb{R}^-$, such that the result is order-isomorphic to $[0, 1]$.



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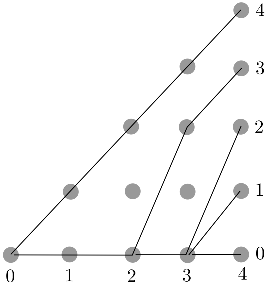

Our procedure:

- We choose the order-type of the $\mathbb{R}^-$-modules, e.g.:

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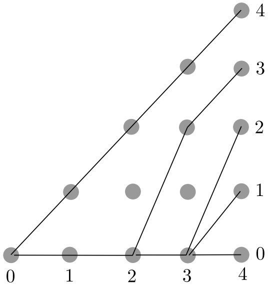


Our procedure:

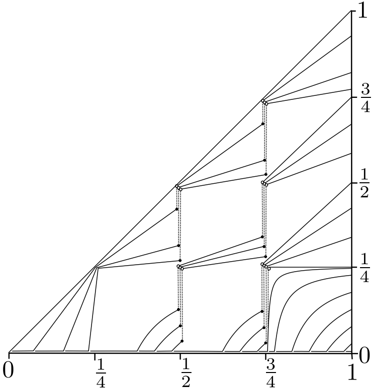
- We choose the order-type of the $\mathbb{R}^-$-modules, e.g.:

$$\mathbb{R}^+,\ \{0\},\ \mathbb{R}^-,\ \mathbb{R}^-,\ \mathbb{R}^-.$$

- We choose the parameters of the homomorphisms between the occurring $\mathbb{R}^-$-modules.

Example of a coextension

The result is a triangular norm:



$$a \odot b \;=\; \begin{cases} a \wedge b & \text{if } a, b > \frac{3}{4}, \\ a \wedge (b - \frac{1}{4}) & \text{if } b > \frac{3}{4} \text{ and } \frac{1}{2} < a \leqslant \frac{3}{4}, \\ a \wedge (b - \frac{1}{2}) & \text{if } b > \frac{3}{4} \text{ and } \frac{1}{4} < a \leqslant \frac{1}{2}, \\ a & \text{if } b > \frac{3}{4},\ a \leqslant \frac{1}{4}, \text{ and } a + b > 1, \\ 0 & \text{if } b > \frac{3}{4},\ a < \frac{1}{4}, \text{ and } a + b \leqslant 1, \\ (a - \frac{1}{4}) & \\ \wedge (b - \frac{1}{4}) \wedge \frac{7}{16} & \text{if } \frac{1}{2} < a, b \leqslant \frac{3}{4}, \\ \frac{1}{8} & \text{if } \frac{5}{8} < b \leqslant \frac{3}{4} \text{ and } \frac{3}{8} < a \leqslant \frac{1}{2}, \\ 0 & \text{if } b \leqslant \frac{3}{4} \text{ and } a \leqslant \frac{3}{8}, \\ & \text{or } b \leqslant \frac{5}{8} \text{ and } a \leqslant \frac{1}{2} \end{cases}$$

Conclusion

Summary

- Cut-continuous pomonoids generalise residuated lattices.
- A tensor product of modules over cut-continuous pomonoids exists.
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Problems and ongoing work

- The universality of the tensor product is restricted to mappings to sup-lattices.

  The plan is elaborate on the case of conditionally complete lattices.
- The shown application to coextensions restricts to the totally ordered case.