On the Characterization of Orthogroups by Disjunctions of Identities

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Every band is a semilattice of rectangular bands.

Theorem (Yamada, 1963)

Every orthodox completely regular semigroup is a semilattice of rectangular groups.

Motivation: Disjunction of Identities

E.S. Lyapin

Atoms in the lattice of identical-inclusion varieties,
Sibirsk. Mat. Zh. **16** (1975), no. 6, 1224-1230.

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Algebraic systems with one action and relation, Leningrad. Gos. Ped. Inst., Leningrad (1985), 21-32.

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R. Thron, J. Koppitz

Free systems of algebras and ultraclosed classes, Acta Mathematica Universitatis Comenianae, Vol. **LXXV**, 1 (2006), 127–136.

R.A.R. Monzo

Semilattices of rectangular bands and groups of order two, http://arxiv.org/ftp/arxiv/papers/1301/1301.0828.pdf (2013) (09.06.2017)

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Y. Shao, M. Ren

On semilattices of rectangular bands and groups, Semigroup Forum **Volume 93** (2016), 201-204.

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Let $X = \{x, y, z, \ldots\}$ be a non-empty set, and let X^+ be the set of all words generated by the elements of X.

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We concentrate on alphabets with two elements, i.e. $X = \{x, y\}$.

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Lemma

For every semigroup S *and every mapping* $h : X \longrightarrow S$ *there is a unique extension of* h *to a homomorphism* $\overline{h} : X^+ \longrightarrow S$.

Disjunction of Identities II

Definition

A set $\sigma \in \mathcal{P}(X^+ \times X^+)$ is said to be a **disjunction of identities** in S (in symbols: $S \models \sigma$) if for all maps $h : X \longrightarrow S$ there is $u \approx v \in \sigma$ with $\overline{h}(u) = \overline{h}(v)$ where we use $u \approx v$ instead of $(u, v) \in X^+ \times X^+$.

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Remark

A set $\Sigma \subseteq \mathcal{P}(X^+ \times X^+)$ holds in S (in symbols: $S \in Mod(\Sigma)$) if $S \models \sigma$ for all $\sigma \in \Sigma$.

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Remark

Within this talk:

1. *Disjunction of identities is called* **identity**.
2. *The elements $u \approx v \in \sigma$ are called* **equations**.

Properties: HS

Lemma (Homomorphic Image)

Let S, T be semigroups, $\Sigma \subseteq \mathcal{P}(X^+ \times X^+)$ be a non-empty set and let $S \in Mod(\Sigma)$. If $\varphi : S \longrightarrow T$ is a homomorphism, then $\varphi(S) \in Mod(\Sigma)$.

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Lemma (Subsemigroup)

Let S be a semigroup, let $\Sigma \subseteq \mathcal{P}(X^{+} \times X^{+})$ be a non-empty set and let $S \in Mod(\Sigma)$. If $T \leq S$, then $T \in Mod(\Sigma)$.

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Remark

Disjunctions of identities are in general not closed under products.

Purpose

Goal

How can we represent particular classes of completely regular semigroups by sets of disjunctions of identities?

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Lemma

Let $P \subseteq \mathbb{N}$ be a non-empty set and let S be a semigroup.
If $S \models \left\{x^{p+1} \approx x : p \in P\right\}$, then S is a completely regular semigroup.

Lemma

Let S be a semilattice Y of completely simple semigroups $S_\alpha (\alpha \in Y)$ and let $S \models \{u_j \approx v_j : j \in J\}$.

General Results

Lemma

Let S be a semilattice Y of completely simple semigroups $S_\alpha (\alpha \in Y)$ and let $S \vDash \{u_j \approx v_j : j \in J\}$.

1. *If leftmost $(u_j) \neq$ leftmost (v_j) for all $j \in J$, then $S_\alpha \in \mathcal{RG}$ for all $\alpha \in Y$.*

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3. *If rightmost $(u_j) \neq$ rightmost (v_j) or leftmost $(u_j) \neq$ leftmost (v_j) for all $j \in J$, then $S_\alpha \in \mathcal{RG} \cup \mathcal{LG}$ for all $\alpha \in Y$.*

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- $\mathcal{L}_p^3$ *class of groups of exponent p,*
- $\mathcal{L}_p^4$ *class of rectangular groups of exponent p (Especially $\mathcal{L}_1^4$ is the class of rectangular bands, $\mathcal{L}_1^4 \subseteq \mathcal{L}_p^4$).*

Back to Motivation

Theorem (Monzo, 2013)

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1. *S is a semilattice Y of semigroups S_α $(\alpha \in Y)$ with $S_\alpha \in \mathcal{L}_1^4 \cup \mathcal{L}_2^3$,*
2. $S \models \{xyx \approx xy^2x, xyx \approx y^2x^2y\}$,
   $S \models \{x^3 \approx x\}$.

Semilattices of particular Orthodox Semigroups

Theorem

Let $P \subset \mathbb{N}$ *be a non-empty set of pairwise coprime numbers, let* $\Gamma = \{1, 2, 3, 4\}$ *and let* $\Pi \in \mathcal{P}(\Gamma \times P)$*. Furthermore, let* $\mathcal{K}$ *be the class of all semigroups* S *such that every* S *is a semilattice* Y *of semigroups* S_α *with* $S_\alpha \in \bigcup_{(\gamma,p)\in\Pi} \mathcal{L}_p^\gamma$ *for* $\alpha \in Y$*. Then there is a non-empty set* $\Sigma \subseteq \mathcal{P}\left(\{x, y\}^+ \times \{x, y\}^+\right)$ *such that* $\mathcal{K} = Mod(\Sigma)$*.*

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Remark (Minimality)

There exists a characterization which is minimal.

Example: Semilattices of Left Groups of Exponent p, Right Groups of Exponent q and Groups of Exponent r

Corollary

Let $p, q, r \in \mathbb{N}$ be pairwise coprime and let S be a semigroup. Then the following statements are equivalent:

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Remark

Observe the finite set for P.

Semilattices of Abelian Groups

Remark

We denote by $\mathcal{A}_p$ *the class of abelian groups of exponent* p *for* $p \in \mathbb{N}$*.*
$(\mathcal{A}_p \subset \mathcal{L}_p^3)$

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Let $P_{\mathcal{A}} \subseteq P \subset \mathbb{N}$ with P a non-empty set of pairwise coprime numbers and let S be a semigroup. Then the following statements are equivalent:

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2. $S \models \{x^{p+1} \approx x : p \in P\}$,
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2. $S \vDash \left\{x \approx x^{p+1}y^p, y \approx y^{p+1}x^p, x \approx y^q x^{q+1}, y \approx x^q y^{q+1}\right\}$.

Thank you for your attention!