On homotopies of algebraic systems

Denis I. Saveliev

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Systems

$\mathfrak{A}$ is an *algebraic system* (shortly: a *system*) iff

$$\mathfrak{A} = (A, F, \ldots, P, \ldots)$$

where A is a set, called the *universe* of $\mathfrak{A}$, and $F, \ldots$ are operations and $P, \ldots$ relations on A. Thus $\mathfrak{A}$ is a model of a first-order language.

$\mathfrak{A}$ is a *universal algebra* iff it contains no P's, thus the language has only functional symbols.

$\mathfrak{A}$ is *finitary* iff it is a model of a first-order finitary language, thus all its F and P are finitary, and *infinitary* otherwise.

An ordinal α is the *arity* of $\mathfrak{A}$ iff

$$\alpha = \sup\{\text{arities of all } F\text{'s and } P\text{'s in } \mathfrak{A}\}.$$

Thus $\alpha \leq \omega$ if $\mathfrak{A}$ is finitary, and any in general.

Homotopies

Operations

Let F and G be β-ary operations on A and B resp.

A $(\beta + 1)$-sequence $(h_i)_{i \leq \beta}$ of maps $h_i : A \to B$ is a *homotopy* between (A, F) and (B, G) iff

$$h_\beta(F(x_i)_{i<\beta}) = G(h_i(x_i))_{i<\beta}$$

for all $x_i \in A$, $i < \beta$.

A homotopy $(h_i)_{i \leq \beta}$ between (A, F) and (B, G) is an *isotopy* iff all the h_i are bijective.

Examples. If F and G are unary, a homotopy is (h_0, h_1) such that $h_1(F(x)) = G(h_0(x))$ for all $x \in A$.
If F and G are binary, a homotopy is (h_0, h_1, h_2) such that $h_2(F(x, y)) = G(h_0(x), h_1(y))$ for all $x, y \in A$.

Relations

Let P and Q be β-ary relations on A and B.

A β-sequence $(h_i)_{i<\beta}$ of maps $h_i : A \to B$ is a *homotopy* between (A, P) and (B, Q) iff

$$P(x_i)_{i<\beta} \text{ implies } Q(h_i(x_i))_{i<\beta}$$

for all $x_i \in A$, $i < \beta$.

A homotopy $(h_i)_{i<\beta}$ between (A, P) and (B, Q) is *strong* iff for all $(y_i)_{i<\beta}$ in B there exist $(x_i)_{i<\beta}$ in A such that

$$Q(y_i)_{i<\beta} \text{ implies } P(x_i)_{i<\beta}$$

and $h_i(x_i) = y_i$ for all $i < \beta$.

A homotopy $(h_i)_{i<\beta}$ between (A, P) and (B, Q) is an *isotopy* iff all the h_i are bijective and

$$P(x_i)_{i<\beta} \text{ iff } Q(h_i(x_i))_{i<\beta}$$

for all $x_i \in A$, $i < \beta$.

In other words, a homotopy is an isotopy iff it is strong and all its components are bijective.

Examples. If P and Q are unary, any homotopy is a homomorphism.
If P and Q are binary, a homotopy is (h_0, h_1) such that $P(x, y)$ implies $Q(h_0(x), h_1(y))$ for all $x, y \in A$.

Systems

Let $\mathfrak{A}$ and $\mathfrak{B}$ be systems in the same language and α their arity.

An α-sequence $(h_i)_{i\leq\alpha}$ of maps $h_i : A \to B$ is a *homotopy* between $\mathfrak{A}$ and $\mathfrak{B}$ iff:

(i) for any operations F and G interpreting the same β-ary functional symbol in $\mathfrak{A}$ and $\mathfrak{B}$ resp., the restriction $(h_i)_{i\leq\beta}$ is a homotopy between (A, F) and (B, G), and
(ii) for any relations P and Q interpreting the same β-ary relational symbol in $\mathfrak{A}$ and $\mathfrak{B}$ resp., the restriction $(h_i)_{i<\beta}$ is a homotopy between (A, P) and (B, Q).

A homotopy between $\mathfrak{A}$ and $\mathfrak{B}$ is *strong* iff so are the restricted homotopies for all relations, and an *isotopy* iff so are the restricted homotopies for all operations and relations.

Remark. Homotopies of systems with many operations or relations can be defined somewhat differently. The results below remain true mutatis mutandis.

Observation. If all h_i coincide, the homotopy (strong homotopy, isotopy) is a homomorphism (strong homomorphism, isomorphism).

Roughly, the smaller is the size of $\{h_i\}_{i \leq \alpha}$ the closer the homotopy is to a homomorphism.

Historical remarks. The concept of isotopy for quasigroups was introduced by Albert (1943), based on his earlier (1942) definition of isotopy for algebras, in turn inspired by Steenrod's work. Albert's initial results:

Any quasigroup is isotopic to a loop,

If a loop is isotopic to a group then they are isomorphic,

were later generalized by Bruck, Hughes, Halaš, Petrescu, and others.

Pseudo-systems

$\mathfrak{A}$ is a *pseudo-algebraic system* (shortly: a *pseudo-system*) in a first-order language of arity α iff

$$\mathfrak{A} = ((A_i)_{i\leq\alpha}, F, \ldots, P, \ldots)$$

where $(A_i)_{i\leq\alpha}$ is an α-sequence of sets, and $F, \ldots$ and $P, \ldots$ are maps and relations such that if F is β-ary then

$$F : \prod_{i<\beta} A_i \to A_\beta$$

and if P is β-ary then

$$P \subseteq \prod_{i<\beta} A_i.$$

If all the A_i coincide, the pseudo-system $\mathfrak{A}$ is (identified with) a system.

Pseudo-subsystems

A pseudo-system $\mathfrak{B} = ((B_i)_{i\leq\alpha}, G, \ldots, Q, \ldots)$ is a *pseudo-subsystem* of a pseudo-system $\mathfrak{A} = ((A_i)_{i\leq\alpha}, F, \ldots, P, \ldots)$ iff:

(i) $B_i \subseteq A_i$ for all $i \leq \alpha$,
(ii) if F and G interpret the same β-ary functional symbol, then

$$G = F \restriction \prod_{i<\beta} B_i,$$

(iii) if P and Q interpret the same β-ary relational symbol, then

$$Q = P \cap \prod_{i<\beta} B_i.$$

Products of pseudo-systems

Let

$$\mathfrak{A}_j = ((A_{j,i})_{i \leq \alpha}, F_j, \ldots, P_j, \ldots)$$

be pseudo-systems in the same language, $j \in J$. Their *product* is the pseudo-system

$$\prod_{j \in J} \mathfrak{A}_j = ((A_i)_{i \leq \alpha}, F, \ldots, P, \ldots) \text{ where:}$$

(i) $A_i = \prod_{j \in J} A_{j,i}$ for all $i \leq \alpha$,
(ii) if F and all the F_j interpret the same β-ary functional symbol, then

$$F(f_i)_{i<\beta} = (F_j(f_i(j))_{i<\beta})_{j \in J}$$

for all f_i in A_i , $i < \beta$,
(iii) if P and all the P_j interpret the same β-ary relational symbol, then

$$P(f_i)_{i<\beta} \text{ iff } P_j(f_i(j))_{i<\beta}) \text{ for all } j \in J$$

for all f_i in A_i , $i < \beta$.

Thus the F and P are defined component-wise.

Example. The product of two pseudo-systems $((A_i)_{i\leq\alpha}, F, \ldots, P, \ldots)$ and $((B_i)_{i\leq\alpha}, G, \ldots, Q, \ldots)$ is

$$((C_i)_{i\leq\alpha}, H, \ldots, R, \ldots)$$

such that:
(i) $C_i = A_i \times B_i$ for all $i \leq \alpha$,
(ii) if H is β-ary then

$$H(x_i, y_i)_{i<\beta} = (F(x_i)_{i<\beta}, G(y_i)_{i<\beta})$$

for all x_i in A_i and y_i in B_i, $i < \beta$,
(iii) if R is β-ary then

$$R(x_i, y_i)_{i<\beta} \text{ iff } P(x_i)_{i<\beta} \text{ and } Q(y_i)_{i<\beta}$$

for all x_i in A_i and y_i in B_i, $i < \beta$.

The following generalizes the standard facts about homomorphisms.

Proposition. *Any homotopy between systems* $\mathfrak{A}$ *and* $\mathfrak{B}$ *forms a pseudo-subsystem of the system* $\mathfrak{A} \times \mathfrak{B}$.

Proposition. *Let* $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ *be pseudo-systems, and let* $\mathfrak{D}$, $\mathfrak{R}$, $\mathfrak{S}$ *be pseudo-subsystems of* $\mathfrak{A}$, $\mathfrak{A} \times \mathfrak{B}$, $\mathfrak{A} \times \mathfrak{C}$ *resp. Then*

$$(R_i \text{``} D_i)_{i \leq \alpha}, \ \ (R_i^{-1})_{i \leq \alpha}, \ \ (R_i \circ S_i)_{i \leq \alpha}$$

form pseudo-subsystems of $\mathfrak{B}$, $\mathfrak{B} \times \mathfrak{A}$, $\mathfrak{B} \times \mathfrak{C}$ *resp.*

Corollary. *Let* $(h_i)_{i \leq \alpha}$ *be a homotopy between* $\mathfrak{A}$ *and* $\mathfrak{B}$. *Then:*

(i) $(h_i \text{``} A)_{i \leq \alpha}$ *forms a pseudo-subsystem of* $\mathfrak{B}$.
(ii) If $(g_i)_{i \leq \alpha}$ *is a homotopy between* $\mathfrak{B}$ *and* $\mathfrak{C}$, *then* $(g_i \circ h_i)_{i \leq \alpha}$ *is a homotopy between* $\mathfrak{A}$ *and* $\mathfrak{C}$.
(iii) If both $(h_i)_{i \leq \alpha}$ *and* $(g_i)_{i \leq \alpha}$ *are isotopies, then so is* $(g_i \circ h_i)_{i \leq \alpha}$.

Quotients of pseudo-systems

A sequence $(\mathfrak{q}_i)_{i\le\alpha}$ of equivalence relations is a *pseudo-congruence* on a pseudo-system $\mathfrak{A}$ iff it forms a pseudo-subsystem of $\mathfrak{A}\times\mathfrak{A}$.

If $\mathfrak{A}=((A_i)_{i\le\alpha},F,\ldots,P,\ldots)$ is a pseudo-system and $(\mathfrak{q}_i)_{i\le\alpha}$ is a pseudo-congruence on it, the *quotient* of $\mathfrak{A}$ by $(\mathfrak{q}_i)_{i\le\alpha}$ is the pseudo-system

$$\mathfrak{A}/(\mathfrak{q}_i)_{i\le\alpha}=((B_i)_{i\le\alpha},G,\ldots,Q,\ldots)$$

where:

(i) $B_i=A_i/\mathfrak{q}_i$ for all $i\le\alpha$,
(ii) if F and G interpret the same β-ary functional symbol, then $G:\prod_{i<\beta}A_i/\mathfrak{q}_i\to A_\beta/\mathfrak{q}_\beta$ is defined by

$$G([x_i]_{\mathfrak{q}_i})_{i<\beta}=[F(x_i)_{i<\beta}]_{\mathfrak{q}_i}$$

for all $x_i\in A_i$, $i<\beta$,
(iii) if P and Q interpret the same β-ary relational symbol, then $Q\subseteq\prod_{i<\beta}A_i/\mathfrak{q}_i$ is defined by

$$Q([x_i]_{\mathfrak{q}_i})_{i<\beta}\ \text{iff}\ P(y_i)_{i<\beta}$$
$$\text{for some } y_i \text{ with } [x_i]_{\mathfrak{q}_i}=[y_i]_{\mathfrak{q}_i},\ i<\beta,$$

for all x_i in A_i, $i<\beta$.

Recall that the *kernel* of a map $f : A \to B$ is the equivalence relation $\ker f = f^{-1} \circ f$ on A, and the *natural map* of an equivalence relation $\mathfrak{q}$ on A is the map $\operatorname{nat} \mathfrak{q} : A \to A/\mathfrak{q}$ defined by $\operatorname{nat} \mathfrak{q}(x) = [x]_{\mathfrak{q}}$ for all x in A.

The *kernel* of a sequence $(f_i)_{i \le \alpha}$ of maps and the *natural sequence* of maps of a sequence $(\mathfrak{q}_i)_{i \le \alpha}$ of equivalences are defined componentwise:

$$\ker(f_i)_{i \le \alpha} = (\ker f_i)_{i \le \alpha},$$

$$\operatorname{nat}(\mathfrak{q}_i)_{i \le \alpha} = (\operatorname{nat} \mathfrak{q}_i)_{i \le \alpha}.$$

The following proposition just shows that the introduced concepts work as expected.

Proposition. *Let $\mathfrak{A}$ be a pseudo-system.*

(i) If $(h_i)_{i\leq\alpha}$ is a homotopy of $\mathfrak{A}$, then its kernel is a pseudo-congruence on $\mathfrak{A}$.
(ii) If $(\mathfrak{q}_i)_{i\leq\alpha}$ is a pseudo-congruence on $\mathfrak{A}$, then its quotient of $\mathfrak{A}$ is a unique pseudo-system $\mathfrak{B}$ on $(A_i/\mathfrak{q}_i)_{i\leq\alpha}$ such that the natural sequence of maps given by the pseudo-congruence is a homotopy between $\mathfrak{A}$ and $\mathfrak{B}$.

This natural sequence will be called the *natural homotopy* between a given pseudo-system and its quotient.

Isotopy theorems

A sequence $(f_i)_{i\leq\alpha}$ of maps is:

(i) *weakly injective* iff all the f_i are injective,
(ii) *strongly injective* iff $\bigcup_{i\leq\alpha} f_i$ is injective,
(iii) *weakly surjective* iff $\bigcup_{i\leq\alpha} f_i$ is surjective,
(iv) *strongly surjective* iff all the f_i are surjective and have the same range,
(v) *uniform* iff all the f_i have the same kernel.

Observation.
1. A homotopy is an isotopy iff it is both strongly injective and strongly surjective.
2. Each homotopy is the (component-wise) composition of a weakly surjective homotopy and an inclusion.

Hence we can concentrate on weakly surjective homotopies.

The following generalizes the First Isomorphism Theorem:

Theorem.
(i) Each weakly surjective homotopy $(h_i)_{i\leq\alpha}$ *is the composition of the natural homotopy given by its kernel and a weakly injective homotopy* $(g_i)_{i\leq\alpha}$ *defined by*

$$g([x_i]_{\ker h_i}) = h_i(x_i)$$

for all x_i *in* A_i *,* $i \leq \alpha$*. Moreover, such a composition is unique.*
(ii) The homotopy $(g_i)_{i\leq\alpha}$ *is an isotopy iff* $(h_i)_{i\leq\alpha}$ *is strongly injective.*
(iii) The natural homotopy $\mathrm{nat}\,(\ker(h_i)_{i\leq\alpha})$ *is a homomorphism iff* $(h_i)_{i\leq\alpha}$ *is uniform.*

Corollary. *A homotopy is the composition of a homomorphim and an isotopy iff it is strongly injective and uniform.*

Analogs of the Second and Third Isomorphism Theorems can obtained in a similar way.

Remark. A version of (a part of) the theorem above, called the First Isotopy Theorem, together with the Second and Third ones, has been announced by Petrescu for finitary universal algebras. He used rather a family of algebras than one pseudo-algebra and formulated these results modulo the isotopy equivalence.

Ultrafilter extensions

The *ultrafilter extension* of a finitary system $\mathfrak{A} = (A, F, \ldots, P, \ldots)$ is the system

$$\widetilde{\mathfrak{A}} = (\boldsymbol{\beta} A, \widetilde{F}, \ldots, \widetilde{P}, \ldots)$$

such that:

(i) $\boldsymbol{\beta} A$ is the set of ultrafilters over A,
(ii) $\widetilde{F}$ is the operation on $\boldsymbol{\beta} A$ such that

$$\widetilde{F}(\mathfrak{u}_1, \ldots, \mathfrak{u}_n) =$$
$$\Big\{A \subseteq Y : (\forall^{\mathfrak{u}_1} x_1) \ldots (\forall^{\mathfrak{u}_n} x_n)\ F(x_1, \ldots, x_n) \in A\Big\}$$

for all $\mathfrak{u}_1, \ldots, \mathfrak{u}_n$ in $\boldsymbol{\beta} A$,
(iii) $\widetilde{P}$ is the relation on $\boldsymbol{\beta} A$ such that

$$\widetilde{R}(\mathfrak{u}_1, \ldots, \mathfrak{u}_n) \text{ iff}$$
$$(\forall^{\mathfrak{u}_1} x_1) \ldots (\forall^{\mathfrak{u}_n} x_n)\ R(x_1, \ldots, x_n)$$

for all $\mathfrak{u}_1, \ldots, \mathfrak{u}_n$ in $\boldsymbol{\beta} A$.

Here $(\forall^{\mathfrak{u}} x)\, \varphi(x)$ means $\{x : \varphi(x)\} \in \mathfrak{u}$.

Theorem. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two systems in the same finitary language, and $(h_i)_{i \le \alpha}$ a homotopy between them. Then its continuous extension $(\widetilde{h_i})_{i \le \alpha}$ is a homotopy between their ultrafilter extensions $\widetilde{\mathfrak{A}}$ and $\widetilde{\mathfrak{B}}$.*

This is also true for strong homotopies and isotopies.

This theorem is a partial case of a stronger result, which generalizes the classical fact of general topology, stating that $\boldsymbol{\beta} A$ is the largest compactification of the discrete space A, to the case when A carries a finitary first-order structure.

Problem. Characterize formulas preserved under isotopies (in a given class of systems).

In quasigroup theory, the properties described by such formulas are called "universal" (not confuse with universal formulas!). E.g. so are the properties of:

- associativity,
- being an Abelian group,

and are not:

- commutativity,
- idempotency,
- having identity.

The same question can be posed for homotopies, strong homotopies, etc.