The Lattice of All Clones Definable by Binary Relations on a Three-Element Set

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What is Studied: Relational Clones

Let R_k be the set of all finitary predicates on $\{0, 1, \ldots, k-1\}$.

Definition. $M \subseteq R_k$ is called a *relational clone* if

1. $\{\mathit{false}^{(0)}, \mathit{true}^{(0)}, \mathit{eq}^{(2)}\} \subseteq M$;
2. and M is closed under *primitive positive formulas*:

$$[\![\; p_1^{(r_1)} \in M, \;\ldots,\; p_n^{(r_n)} \in M,$$

$$p^{(r)}(x_1, \ldots, x_r) = \exists y_1 \ldots \exists y_s \Big(p_1^{(r_1)}(z_{1,1}, \ldots, z_{1,r_1}) \wedge \ldots \wedge p_n^{(r_n)}(z_{n,1}, \ldots, z_{n,r_n}) \Big)$$

$$]\!] \Longrightarrow p^{(r)} \in M$$

($s \geqslant 0$, $n \geqslant 0$; $z_{k,j}$ is one of the symbols: "x_1", ..., "x_r", "y_1", ..., "y_s").

What is Studied: Closure Operator for Predicates

Definition. We say that p *is generated from* M if $p \in \{false^{(0)}, true^{(0)}, eq^{(2)}\}$ or if p can be expressed as a primitive positive formula:

$$p^{(r)}(x_1, \ldots, x_r) =$$
$$= \exists y_1 \ldots \exists y_s \left(p_1^{(r_1)}(z_{1,1}, \ldots, z_{1,r_1}) \wedge \ldots \wedge p_n^{(r_n)}(z_{n,1}, \ldots, z_{n,r_n}) \right)$$

The set of all predicates generated from M we denote as

$$[M]_{p.p.}$$

What is Studied: Finitely Generated Relational Clones

Definition. A relational clone M is *finitely generated* if

$$\exists M_0 \subseteq M \colon \big(M_0 \textit{ is finite}\big) \wedge \big(M = [M_0]_{p.p.}\big).$$

Definition. *Degree* of a (finitely generated) relational clone M is the smallest number r such that

$$M = \big[M \cap R_k(r)\big]_{p.p.}$$

where $R_k(r)$ is the set of all predicates of arity r on $\{0, 1, \dots, k-1\}$.

What is Studied: Finitely Generated Relational Clones

Number of relational clones of degree $\leqslant r$ on k-element set.

	$r = 2$	$r = 3$	$r = 4$
$k = 2$	19	44	54
$k = 3$	?	$> 2^{50}$	$> 2^{500}$
$k = 4$	$> 2^{60}$	$> 2^{2^{20}}$	$> 2^{2^{70}}$

[1] Figures for $k = 2$ are known from *Блохина Г. Н. О предикатном описании классов Поста. Дискретный анализ 16, 16–29 (1970).*

[2] Figures in red are known from *Dmitriy Zhuk, Stanislav Moiseev. On the Clones Containing a Near-Unanimity Function.* ISMVL 2013: 129-134.

Our Aim

Construct all relational clones of degree $\leqslant 2$ ***on three-element set:***

$$\left\{ M \mid M = [M \cap R_3(2)]_{p.p.} \right\} = ?$$

Our Tools and Obstacles

- **Main tool**

  Generation of predicates.

- **Main obstacle**

  There are infinitely many primitive positive formulas

$$p(x_1, \dots, x_r) = \exists y_1 \dots \exists y_s \ \varphi(\dots)$$

  - No limit for arities of predicates.
  - No limit for the number of existential quantifiers.

- **Our solution**
  - Consider binary predicates only.
  - Limit the complexity of formulas.

Solution: Weak Closure Operator

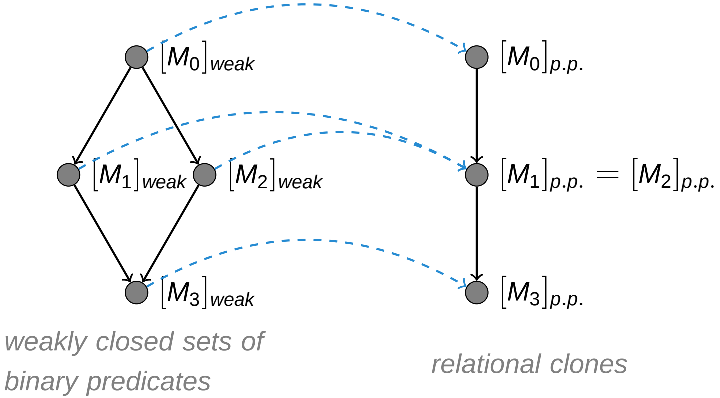
Definition Let $M \subseteq R_k(2)$. By $[M]_{weak}$ we denote the smallest set satisfying the axioms:

1. $\{false^{(2)}, true^{(2)}, eq^{(2)}\} \subseteq [M]_{weak}$.
2. If $p \in M$ then $p \in [M]_{weak}$.
3. $[M]_{weak}$ is closed under the following operations:

$$op\_permutation(p)(x_0, x_1) \equiv p(x_1, x_0)$$
$$op\_conjunction(p_1, p_2)(x_0, x_1) \equiv p_1(x_0, x_1) \wedge p_2(x_0, x_1)$$
$$op\_composition(p_1, p_2)(x_0, x_1) \equiv \exists y \left(p_1(x_0, y) \wedge p_2(y, x_1)\right)$$

Our Plan to Compute All Relational Clones

1. Compute all $[\ ]_{weak}$-closed sets of binary predicates.
2. Analyze the connection between weakly closed sets of binary predicates and relational clones.



Step 1: Computation of All Weakly Closed Sets of Binary Predicates

Algorithm

- Fix a sequence of all predicates:

$$p_1, p_2, \dots, p_{512}.$$

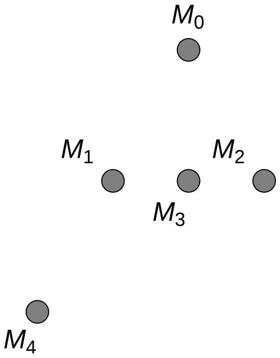
- We will use an inductive procedure with 512 steps.

Algorithm: Induction Basis

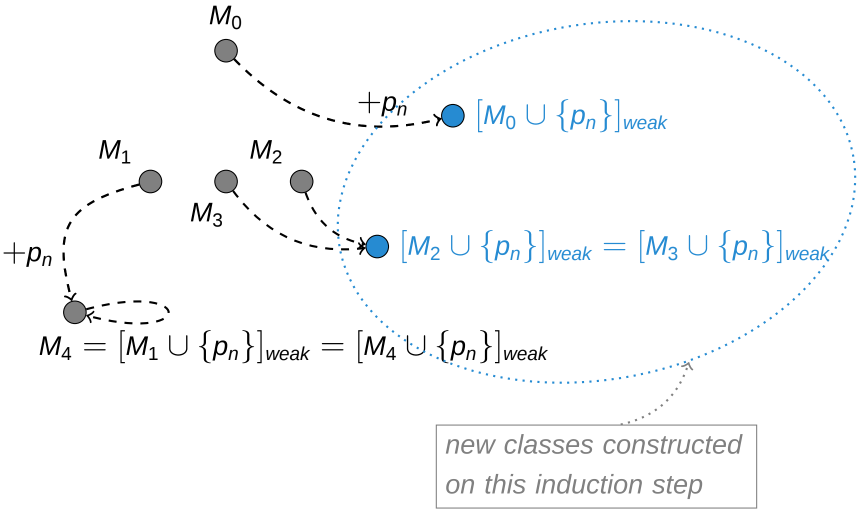
Algorithm: Induction Step n

Given: A set $\{M_0, \ldots, M_{s_{n-1}}\}$ of all $[\ ]_{weak}$-closed predicate sets generated by predicates $p_1, p_2, \ldots, p_{n-1}$.

Task: Construct all $[\ ]_{weak}$-closed predicate sets generated by $p_1, p_2, \ldots, p_{n-1}, p_n$.



Algorithm: Induction Step n



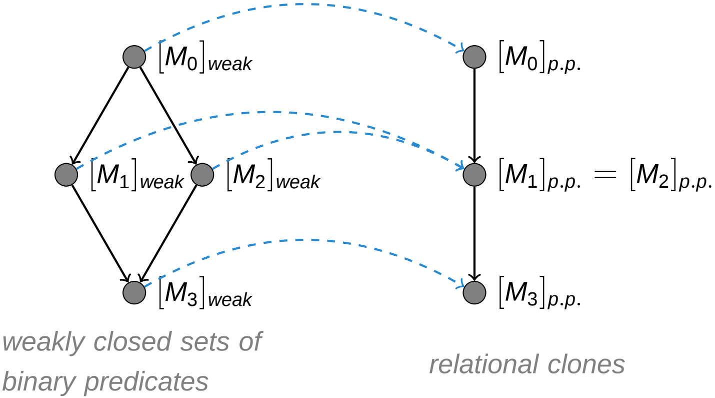
Results

Theorem

On three-element set, there are exactly **2**, **079**, **040** weakly closed sets of binary predicates.

Our Plan to Compute All Relational Clones

1. Compute all $[\ ]_{weak}$-closed sets of predicates.
2. Analyze the connection between weakly closed sets of binary predicates and relational clones.

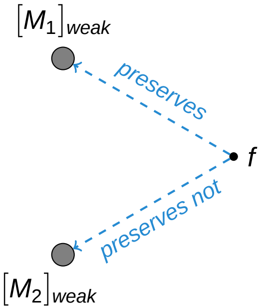


Step 2: Clone Separation

Clone separation problem

Given: $[M_1]_{weak} \neq [M_2]_{weak}$

Decide: $[M_1]_{p.p.} \stackrel{?}{=} [M_2]_{p.p.}$



Step 2: Clone Separation

Microsoft Z3 Solver

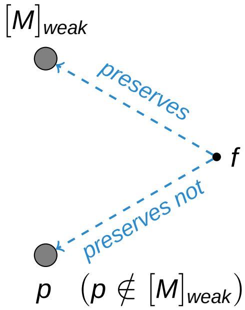
- Z3 is a Satisfiability Modulo Theories (SMT) solver.
- Z3 integrates several decision procedures.
- Z3 is used in several program analysis, verification, test case generation projects at Microsoft.

`https://github.com/Z3Prover/z3`

Clone Separation: Specification for Z3

```
;; define a finite domain
(declare-datatypes () ((E3 V0 V1 V2)))

;; define a predicate ‘p1‘
(declare-fun p1 (E3 E3) Bool)
(assert (= (p1 V0 V0) true))
(assert (= (p1 V0 V1) true))
(assert (= (p1 V0 V2) true))
(assert (= (p1 V1 V0) false))
(assert (= (p1 V1 V1) true))
(assert (= (p1 V1 V2) true))
(assert (= (p1 V2 V0) false))
(assert (= (p1 V2 V1) false))
(assert (= (p1 V2 V2) true))
```



Clone Separation: Specification for Z3

```
;; declare ‘f‘ as an uninterpreted functional symbol
(declare-fun f (E3 E3) E3)

;; ‘f‘ preserves ‘p1‘
(assert (forall ((x0_0 E3) (x0_1 E3) (x1_0 E3) (x1_1 E3))
          (implies (and (p1 x0_0 x0_1) (p1 x1_0 x1_1))
                   (and (p1 (f x0_0 x1_0) (f x0_1 x1_1))))))

;; ‘f‘ does not preserve ‘p‘
(assert (not (forall ((x0_0 E3) (x0_1 E3) (x1_0 E3) (x1_1 E3))
               (implies (and (p x0_0 x0_1) (p x1_0 x1_1))
                        (and (p (f x0_0 x1_0) (f x0_1 x1_1)))))))
```

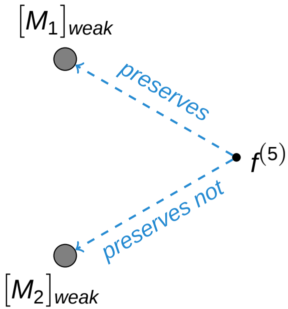
Results

Theorem *(on clone separation)*

Given $[M_1]_{weak} \neq [M_2]_{weak}$.

There exists a function $f^{(5)}$ of arity 5 such that

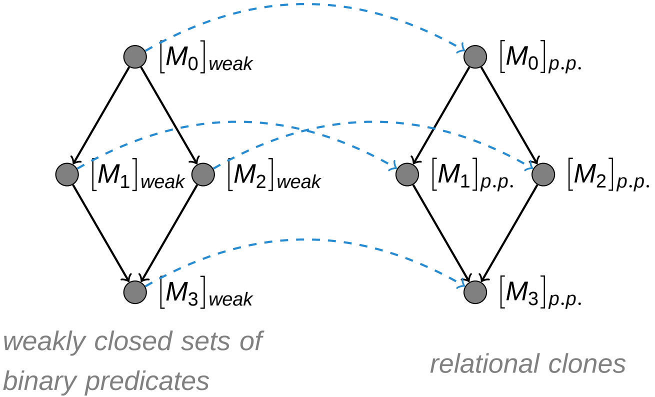
$$(f^{(5)} \text{ preserves } [M_1]_{weak}) \wedge \neg(f^{(5)} \text{ preserves } [M_2]_{weak}).$$



Results

Corollary

Lattice of all weakly closed sets of binary predicates is isomorphic to the lattice of all relational clones of degree $\leqslant 2$.



Results

Theorem *(on the closure operator in $R_3(2)$)*

Let $M \subseteq R_3(2)$ such that $\{\mathit{false}^{(2)}, \mathit{true}^{(2)}, \mathit{eq}^{(2)}\} \subseteq M$.

Then

$$[M]_{p.p.} \cap R_3(2) = [M]_{weak}.$$

Example

- p is expressed as a primitive positive formula over p_1, p_2, p_3, p_4, p_5:

$$p(x_1, x_2) = \exists y_1\ \exists y_2\ \Big(p_1(x_1, y_1) \wedge p_2(x_2, y_1) \wedge \\ \wedge p_3(y_1, y_2) \wedge p_4(y_2, x_1) \wedge p_5(y_2, x_2)\Big).$$

- Fix $p_1, p_2, p_3, p_4, p_5 \in R_3(2)$. Then there exists a formula φ over

$$op\_permutation(p)(x_0, x_1) \equiv p(x_1, x_0)$$

$$op\_conjunction(p_1, p_2)(x_0, x_1) \equiv p_1(x_0, x_1) \wedge p_2(x_0, x_1)$$

$$op\_composition(p_1, p_2)(x_0, x_1) \equiv \exists y\ \big(p_1(x_0, y) \wedge p_2(y, x_1)\big)$$

such that

$$p(x_1, x_2) \sim \varphi(false^{(2)}, true^{(2)}, eq^{(2)}, p_1, p_2, p_3, p_4, p_5)(x_1, x_2).$$

- φ depends on p_1, p_2, p_3, p_4, p_5.

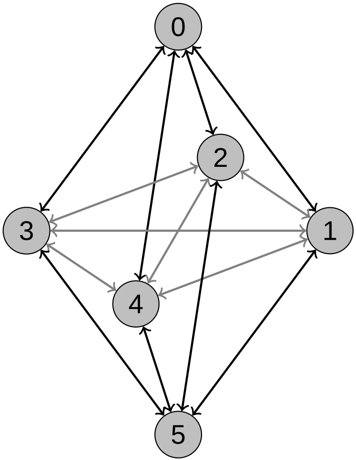
On Weak Closure for $k \geqslant 4$

Observation *(known from Dmitriy N. Zhuk, Reinhard Pöschel)*

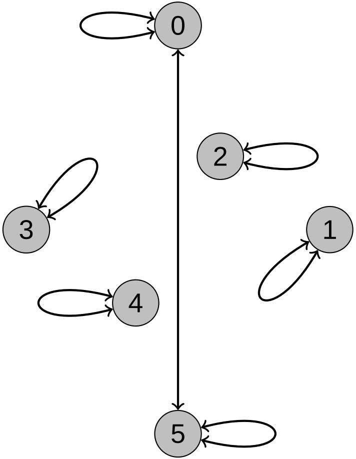
Let $k \geqslant 4$. There exists $M \subseteq R_k(2)$ such that

$$[M]_{p.p.} \cap R_k(2) \neq [M]_{weak}.$$

On Weak Closure for $k \geqslant 4$



(a) Predicate $p \in R_6(2)$

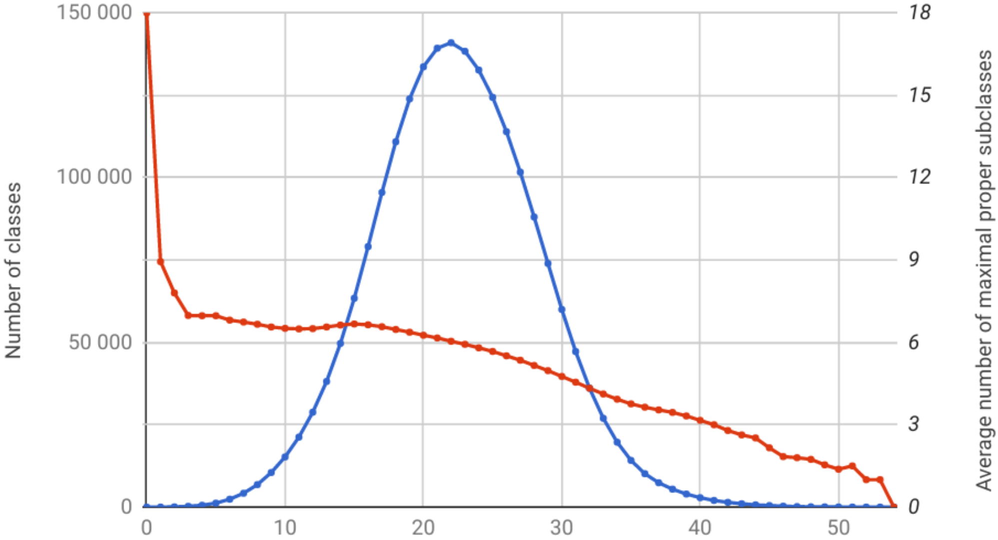


(b) Predicate $q \in R_6(2)$

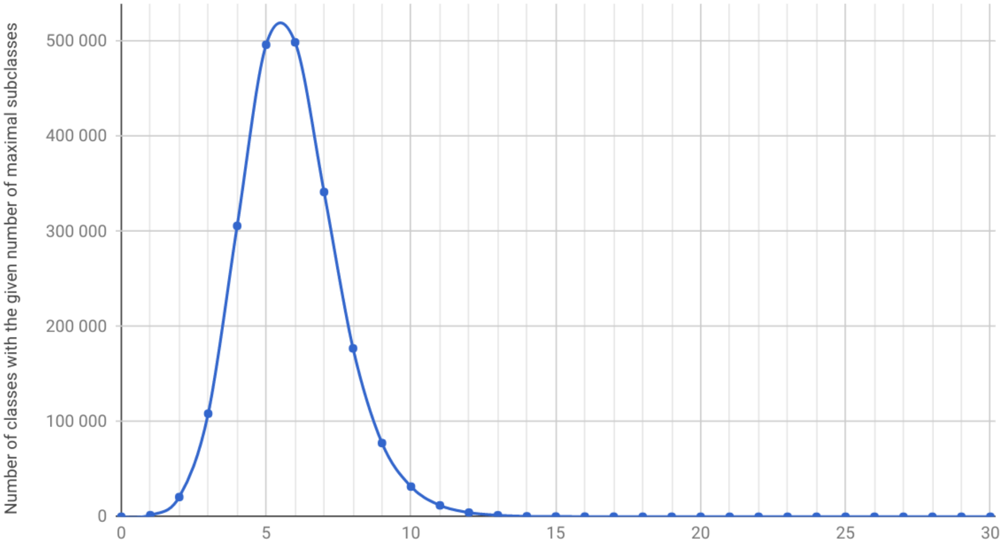
On Weak Closure for $k \geqslant 4$

- $M = \{false^{(2)}, true^{(2)}, eq^{(2)}, p\}$.
- $[M]_{weak} = M$.
- $q(x_1, x_2) = \exists y_1\, \exists y_2 \ldots \exists y_{k-2} \left(\bigwedge_{i=1}^{k-2} p(x_1, y_i) \wedge \bigwedge_{i,j:\ i \neq j} p(y_i, y_j) \wedge \bigwedge_{j=1}^{k-2} p(y_j, x_2) \right)$.
- $q \in [M]_{p.p.} \setminus [M]_{weak}$.

Results


150 000
100 000
50 000
0
Number of classes
0
10
20
30
40
50
18
15
12
9
6
3
0
Average number of maximal proper subclasses

Results


Number of classes with the given number of maximal subclasses
500 000
400 000
300 000
200 000
100 000
0
0
5
10
15
20
25
30

Summary: Finitely Generated Relational Clones

Number of relational clones of degree $\leqslant r$ on k-element set.

	$r = 2$	$r = 3$	$r = 4$
$k = 2$	19	44	54
$k = 3$	$2,079,040$	$> 2^{50}$	$> 2^{500}$
$k = 4$	$> 2^{60}$	$> 2^{2^{20}}$	$> 2^{2^{70}}$

Appendix

History: Clones of Degree $\leqslant 2$

Bashtanov (1989)

Classification of binary predicates in P_3 modulo automorphisms of P_3 and permutation of variables.

- All 74 equivalence classes were found.
- The number of corresponding clones of functions is 67.

History: Clones of Degree $\leqslant 2$

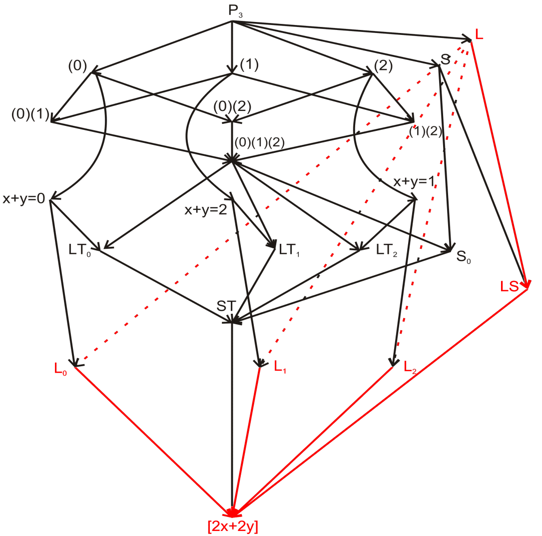
Marchenkov S. S. (1997, 2003).

A description of clones in P_3 containing the ternary discriminator function

$$p(x, y, z) = \begin{cases} z & \text{if } x = y; \\ x & \text{otherwise}. \end{cases}$$

144 clones were described, 6 were missed.

History: Clones of Degree $\leqslant 2$



Rosenberg (1993).

A description of all 23 clones in P_3 containing $f(x, y) = 2x + 2y$.

History: Computer Calculations in Clone Theory

Theorem [Wilde, Raney, 1972, without proof]

Let

$$(f \circ g)(x) \equiv f(g(x))$$

Then $(P_3(1), \circ)$ *has exactly*

$$1299$$

subsemigroups (including $\varnothing$*).*

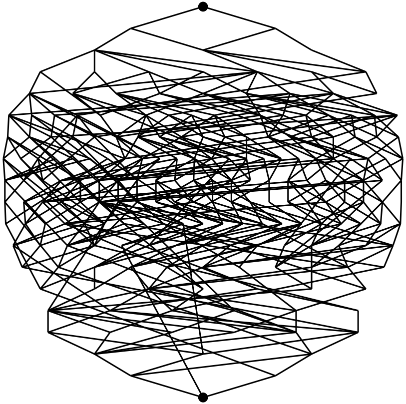
History: Computer Calculations in Clone Theory

Theorem [Csákany, 1983, without proof]

P_3 has exactly 84 minimal clones.

Thank You!

$\{false^{(0)}, true^{(0)}, eq^{(2)}\}$



R_3

$$= \quad 2{,}079{,}040$$

What is Studied?

Let P_k be the set of all finitary functions on $\{0, 1, \ldots, k-1\}$ $(k < \aleph_0)$.

Definition. $F \subseteq P_k$ is called a *clone* if

1. F contains all the projection mappings:

$$pr_j^{(r)} : (x_1, \ldots, x_r) \mapsto x_j$$

2. and F is closed under superposition of functions:

$$\llbracket \; f^{(n)} \in F, \; f_1^{(r_1)} \in F, \; \ldots, \; f_n^{(r_n)} \in F,$$

$$h^{(r)}(x_1, \ldots, x_r) = f^{(n)}\Big(f_1^{(r)}(x_1, \ldots, x_r), \ldots, f_n^{(r)}(x_1, \ldots, x_r)\Big)$$

$$\rrbracket \implies h^{(r)} \in F.$$

What is Studied?

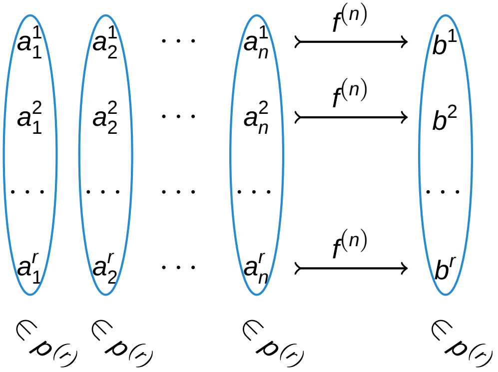
- The set of all clones of P_k forms a lattice $\mathbb{L}(P_k)$ with respect to inclusion $\subseteq$.
- The lattice $\mathbb{L}(P_k)$ is the main object of study in clone theory.
  1. $\mathbb{L}(P_1)$ contains exactly one element.
  2. $\mathbb{L}(P_2)$ is countable.
  3. $\mathbb{L}(P_k)$ ($k \geqslant 3$) has the cardinality of the continuum.

Main Tools: Idea of a Function Preserving a Relation

Let R_k be the set of all finitary predicates on $\{0, 1, \ldots, k-1\}$.

Definition. $f^{(n)} \in P_k$ *preserves* $p^{(r)} \in R_k$ if

$$[\![\, a_1 \in p^{(r)},\ \ldots,\ a_n \in p^{(r)}]\!] \Longrightarrow f^{(n)}(a_1, \ldots, a_n) \in p^{(r)}.$$



Main Tools: Idea of a Function Preserving a Relation

Rosenberg (1970)

Description of all maximal clones in P_k based on the idea of a function preserving a relation.

Theorem [Rosenberg]

Every maximal clone in P_k has the form

$$\mathit{Pol}\ \rho \equiv \{ f \in P_k \mid f \text{ preserves } \rho \}$$

where $\rho \in R_k^{(r)}$ $(1 \leqslant r \leqslant k)$.

Main Tools: The Galois Connection Between Clones and Relational Clones

Krasner (1939, 1945, 1968/69)

The connection between algebras and relations was introduced. A special Galois connection between subsets of the permutation groups

$$S_n \equiv \{f \in P_k(1) \mid f \text{ is bijective}\}$$

and subsets of R_k was formulated.

Main Tools: The Galois Connection Between Clones and Relational Clones

Geiger (1968); Bodnarchuk et al. (1969)

Galois theory for function algebras and relation algebra.

Theorem [Geiger; Bodnarchuk et al.]

There exists a one-to-one correspondence between clones and relational clones.

- If $F \subseteq P_k$ is a clone, then

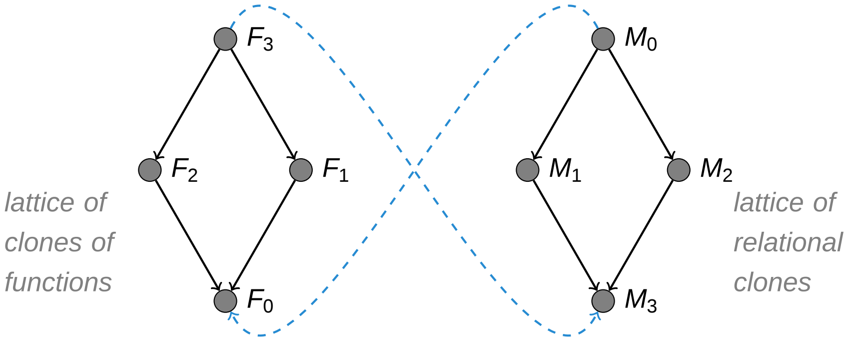
$$Pol\ Inv\ F = F.$$

- If $M \subseteq R_k$ is a relational clone, then

$$Inv\ Pol\ M = M.$$

Main Tools: The Galois Connection Between Clones and Relational Clones

$Inv\colon F \mapsto \{p \in R_k \mid \forall f \in F : f \text{ preserves } p\}$



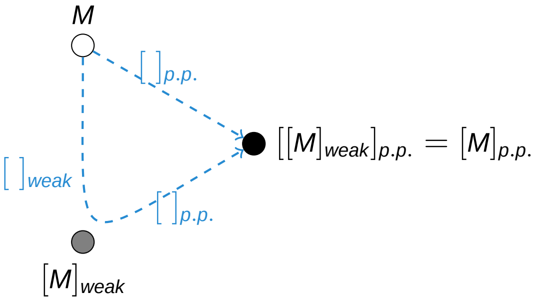
$Pol\colon M \mapsto \{f \in P_k \mid \forall p \in M : f \text{ preserves } p\}$

Solution: Weak Closure Operator

Remark *(on consistency of weak closure operator)*

Weak closure $[\ ]_{weak}$ is consistent with the closure under all primitive positive formulas:

$$[[M]_{weak}]_{p.p.f} = [M]_{p.p.}.$$



Appendix

Some properties of the lattice of all relational clones of degree $\leqslant 2$.

Total number of classes	$2,079,040$
Number of maximal subclasses	18
The longest chain size	55
Number of maximal subclones of a given clone	0—30 (average $\approx$ 5.8)