The Lattice of All Clones
Definable by Binary Relations
on a Three-Element Set

Stanislav V. Moiseev
<stanislav.moiseev@gmail.com>

The 94th Workshop on General Algebra (AAA94)
Novi Sad, Serbia, June 15–18, 2017
What is Studied: Relational Clones

Let $R_k$ be the set of all finitary predicates on $\{0, 1, \ldots, k - 1\}$.

**Definition.** $M \subseteq R_k$ is called a relational clone if

1. $\{false^{(0)}, true^{(0)}, eq^{(2)}\} \subseteq M$;
2. and $M$ is closed under primitive positive formulas:

$$\left[ p^{(r_1)}_1 \in M, \ldots, p^{(r_n)}_n \in M, \right. $$
$$p^{(r)}(x_1, \ldots, x_r) = \exists y_1 \ldots \exists y_s \left( p^{(r_1)}_1(z_{1,1}, \ldots, z_{1,r_1}) \wedge \ldots \wedge p^{(r_n)}_n(z_{n,1}, \ldots, z_{n,r_n}) \right) $$
$$\left. \implies p^{(r)} \in M \right]$$

$(s \geq 0, n \geq 0; z_{k,j}$ is one of the symbols: $"x_1", \ldots, "x_r", "y_1", \ldots, "y_s" )$. 
What is Studied: Closure Operator for Predicates

**Definition.** We say that $p$ is generated from $M$ if $p \in \{\text{false}^{(0)}, \text{true}^{(0)}, \text{eq}^{(2)}\}$ or if $p$ can be expressed as a primitive positive formula:

$$p^{(r)}(x_1, \ldots, x_r) = \exists y_1 \ldots \exists y_s \left( p^{(r_1)}_1(z_{1,1}, \ldots, z_{1,r_1}) \land \ldots \land p^{(r_n)}_n(z_{n,1}, \ldots, z_{n,r_n}) \right)$$

The set of all predicates generated from $M$ we denote as:

$$[M]_{p,p}$$
What is Studied: Finitely Generated Relational Clones

**Definition.** A relational clone $M$ is *finitely generated* if

$$\exists M_0 \subseteq M: (M_0 \text{ is finite}) \land (M = [M_0]_{p,p}).$$

**Definition.** *Degree* of a (finitely generated) relational clone $M$ is the smallest number $r$ such that

$$M = [M \cap R_k(r)]_{p,p}.$$

where $R_k(r)$ is the set of all predicates of arity $r$ on $\{0, 1, \ldots, k - 1\}$. 

What is Studied: Finitely Generated Relational Clones

Number of relational clones of degree \( \leq r \) on \( k \)-element set.

<table>
<thead>
<tr>
<th></th>
<th>( r = 2 )</th>
<th>( r = 3 )</th>
<th>( r = 4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k = 2 )</td>
<td>19</td>
<td>44</td>
<td>54</td>
</tr>
<tr>
<td>( k = 3 )</td>
<td>?</td>
<td>( &gt; 2^{50} )</td>
<td>( &gt; 2^{500} )</td>
</tr>
<tr>
<td>( k = 4 )</td>
<td>( &gt; 2^{60} )</td>
<td>( &gt; 2^{220} )</td>
<td>( &gt; 2^{270} )</td>
</tr>
</tbody>
</table>

1. Figures for \( k = 2 \) are known from Блохина Г. Н. О предикатном описании классов Поста. Дискретный анализ 16, 16–29 (1970).

2. Figures in red are known from Dmitriy Zhuk, Stanislav Moiseev. On the Clones Containing a Near-Unanimity Function. ISMVL 2013: 129-134.
Our Aim

Construct all relational clones of degree $\leq 2$ on three-element set:

\[ \{ M \mid M = [M \cap R_3(2)]_{p,p.} \} = ? \]
Our Tools and Obstacles

- **Main tool**
  
  Generation of predicates.

- **Main obstacle**
  
  There are infinitely many primitive positive formulas

  \[ p(x_1, \ldots, x_r) = \exists y_1 \ldots \exists y_s \varphi(\ldots) \]

  - No limit for arities of predicates.
  - No limit for the number of existential quantifiers.

- **Our solution**
  
  - Consider binary predicates only.
  - Limit the complexity of formulas.
Solution: Weak Closure Operator

**Definition** Let $M \subseteq R_k(2)$. By $[M]_{weak}$ we denote the smallest set satisfying the axioms:

1. $\{false^{(2)}, true^{(2)}, eq^{(2)}\} \subseteq [M]_{weak}$.
2. If $p \in M$ then $p \in [M]_{weak}$.
3. $[M]_{weak}$ is closed under the following operations:

   - $op\_permutation(p)(x_0, x_1) \equiv p(x_1, x_0)$
   - $op\_conjunction(p_1, p_2)(x_0, x_1) \equiv p_1(x_0, x_1) \land p_2(x_0, x_1)$
   - $op\_composition(p_1, p_2)(x_0, x_1) \equiv \exists y \left( p_1(x_0, y) \land p_2(y, x_1) \right)$
Our Plan to Compute All Relational Clones

1. Compute all $[\cdot]_{\text{weak}}$-closed sets of binary predicates.

2. Analyze the connection between weakly closed sets of binary predicates and relational clones.
Step 1: Computation of All Weakly Closed Sets of Binary Predicates

Algorithm

- Fix a sequence of all predicates:
  \[ p_1, p_2, \ldots, p_{512} \]

- We will use an inductive procedure with 512 steps.
Algorithm: Induction Basis

[∅]_{weak}
Given: A set \( \{M_0, \ldots, M_{s_n-1}\} \) of all weak-closed predicate sets generated by predicates \( p_1, p_2, \ldots, p_{n-1} \).

Task: Construct all weak-closed predicate sets generated by \( p_1, p_2, \ldots, p_{n-1}, p_n \).
Algorithm: Induction Step $n$

\begin{align*}
M_4 &= [M_1 \cup \{p_n\}]_{\text{weak}} = [M_4 \cup \{p_n\}]_{\text{weak}} \\
\end{align*}

new classes constructed on this induction step
Results

Theorem

On three-element set, there are exactly 2,079,040 weakly closed sets of binary predicates.
1. Compute all \([ \ ]_{weak}\)-closed sets of predicates.

2. Analyze the connection between weakly closed sets of binary predicates and relational clones.
Clone separation problem

**Given:** \([M_1]_{\text{weak}} \neq [M_2]_{\text{weak}}\)

**Decide:** \([M_1]_{p.p.} \equiv [M_2]_{p.p.}\)
Step 2: Clone Separation

Microsoft Z3 Solver

- Z3 is a Satisfiability Modulo Theories (SMT) solver.
- Z3 integrates several decision procedures.
- Z3 is used in several program analysis, verification, test case generation projects at Microsoft.

https://github.com/Z3Prover/z3
Clone Separation: Specification for Z3

```plaintext
;; define a finite domain
(declare-datatypes () ((E3 V0 V1 V2)))

;; define a predicate 'p1'
(declare-fun p1 (E3 E3) Bool)
(assert (= (p1 V0 V0) true))
(assert (= (p1 V0 V1) true))
(assert (= (p1 V0 V2) true))
(assert (= (p1 V1 V0) false))
(assert (= (p1 V1 V1) true))
(assert (= (p1 V1 V2) true))
(assert (= (p1 V2 V0) false))
(assert (= (p1 V2 V1) false))
(assert (= (p1 V2 V2) true))
```

\[ [M]_{\text{weak}} \]

\( p \quad (p \not\in [M]_{\text{weak}}) \)

- preserves
- preserves not
Clone Separation: Specification for Z3

;;; declare ‘f’ as an uninterpreted functional symbol
(declare-fun f (E3 E3) E3)

;;; ‘f’ preserves ‘p1’
(assert (forall ((x0_0 E3) (x0_1 E3) (x1_0 E3) (x1_1 E3))
    (implies (and (p1 x0_0 x0_1) (p1 x1_0 x1_1))
    (and (p1 (f x0_0 x1_0) (f x0_1 x1_1))))))

;;; ‘f’ does not preserve ‘p’
(assert (not (forall ((x0_0 E3) (x0_1 E3) (x1_0 E3) (x1_1 E3))
    (implies (and (p x0_0 x0_1) (p x1_0 x1_1))
    (and (p (f x0_0 x1_0) (f x0_1 x1_1)))))))
Results

Theorem (on clone separation)

Given $[M_1]_{\text{weak}} \neq [M_2]_{\text{weak}}$.

There exists a function $f^{(5)}$ of arity 5 such that

$$ (f^{(5)} \text{ preserves } [M_1]_{\text{weak}}) \land \neg (f^{(5)} \text{ preserves } [M_2]_{\text{weak}}). $$
Corollary

Lattice of all weakly closed sets of binary predicates is isomorphic to the lattice of all relational clones of degree \( \leq 2 \).
Theorem (on the closure operator in $R_3(2)$)

Let $M \subseteq R_3(2)$ such that \{false$^{(2)}$, true$^{(2)}$, eq$^{(2)}$\} $\subseteq M$.

Then

$$[M]_{p.p.} \cap R_3(2) = [M]_{weak}.$$
Example

- $p$ is expressed as a primitive positive formula over $p_1, p_2, p_3, p_4, p_5$:

$$p(x_1, x_2) = \exists y_1 \exists y_2 \left( p_1(x_1, y_1) \land p_2(x_2, y_1) \land \right.$$  
$$\left. \land p_3(y_1, y_2) \land p_4(y_2, x_1) \land p_5(y_2, x_2) \right).$$

- Fix $p_1, p_2, p_3, p_4, p_5 \in R_3(2)$. Then there exists a formula $\varphi$ over

$$\text{op_permutation}(p)(x_0, x_1) \equiv p(x_1, x_0)$$

$$\text{op_conjunction}(p_1, p_2)(x_0, x_1) \equiv p_1(x_0, x_1) \land p_2(x_0, x_1)$$

$$\text{op_composition}(p_1, p_2)(x_0, x_1) \equiv \exists y \left( p_1(x_0, y) \land p_2(y, x_1) \right)$$

such that

$$p(x_1, x_2) \sim \varphi(\text{false}^{(2)}, \text{true}^{(2)}, \text{eq}^{(2)}; p_1, p_2, p_3, p_4, p_5)(x_1, x_2).$$

- $\varphi$ depends on $p_1, p_2, p_3, p_4, p_5$. 
Observation \textit{(known from Dmitriy N. Zhuk, Reinhard Pöschel)}

Let \( k \geq 4 \). There exists \( M \subseteq R_k(2) \) such that

\[
[M]_{p.p.} \cap R_k(2) \neq [M]_{\text{weak}}.
\]
On Weak Closure for $k \geq 4$

(a) Predicate $p \in R_6(2)$

(b) Predicate $q \in R_6(2)$
On Weak Closure for $k \geq 4$

- $M = \{\text{false}^{(2)}, \text{true}^{(2)}, \text{eq}^{(2)}, p\}$.

- $[M]_{\text{weak}} = M$.

- $q(x_1, x_2) = \exists y_1 \exists y_2 \ldots \exists y_{k-2} \left( \bigwedge_{i=1}^{k-2} p(x_1, y_i) \land \bigwedge_{i,j \colon i \neq j} p(y_i, y_j) \land \bigwedge_{j=1}^{k-2} p(y_j, x_2) \right)$.

- $q \in [M]_{p.p.} \setminus [M]_{\text{weak}}$. 
Results
Results

Number of classes with the given number of maximal subclasses
Summary: Finitely Generated Relational Clones

Number of relational clones of degree $\leq r$ on $k$-element set.

<table>
<thead>
<tr>
<th></th>
<th>$r = 2$</th>
<th>$r = 3$</th>
<th>$r = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k = 2$</td>
<td>19</td>
<td>44</td>
<td>54</td>
</tr>
<tr>
<td>$k = 3$</td>
<td>$2,079,040$</td>
<td>$&gt; 2^{50}$</td>
<td>$&gt; 2^{500}$</td>
</tr>
<tr>
<td>$k = 4$</td>
<td>$&gt; 2^{60}$</td>
<td>$&gt; 2^{220}$</td>
<td>$&gt; 2^{270}$</td>
</tr>
</tbody>
</table>
Bashtanov (1989)

Classification of binary predicates in $P_3$ modulo automorphisms of $P_3$ and permutation of variables.

- All 74 equivalence classes were found.
- The number of corresponding clones of functions is 67.

A description of clones in $P_3$ containing the ternary discriminator function

$$p(x, y, z) = \begin{cases} 
  z & \text{if } x = y; \\
  x & \text{otherwise.}
\end{cases}$$

144 clones were described, 6 were missed.
A description of all 23 clones in $P_3$ containing $f(x, y) = 2x + 2y$. 

Rosenberg (1993). 

History: Clones of Degree $\leq 2$
Theorem [Wilde, Raney, 1972, without proof]

Let

\[(f \circ g)(x) \equiv f(g(x))\]

Then \((P_3(1), \circ)\) has exactly 1299 subsemigroups (including \(\emptyset\)).
Theorem [Csákany, 1983, without proof]

$P_3$ has exactly 84 minimal clones.
Thank You!

\[ \{\text{false}^{(0)}, \text{true}^{(0)}, \text{eq}^{(2)}\} \]

\[ R_3 = 2,079,040 \]
What is Studied?

Let $P_k$ be the set of all finitary functions on $\{0, 1, \ldots, k-1\}$ ($k < \aleph_0$).

**Definition.** $F \subseteq P_k$ is called a *clone* if

1. $F$ contains all the projection mappings:
   
   $$pr_j^{(r)} : (x_1, \ldots, x_r) \mapsto x_j$$

2. and $F$ is closed under superposition of functions:
   
   \[
   \begin{align*}
   &\quad \left[ f^{(n)} \in F, \ f_1^{(r_1)} \in F, \ldots, f_n^{(r_n)} \in F, \right. \\
   &\quad \left. h^{(r)}(x_1, \ldots, x_r) = f^{(n)}\left(f_1^{(r)}(x_1, \ldots, x_r), \ldots, f_n^{(r)}(x_1, \ldots, x_r)\right) \right] \implies h^{(r)} \in F.
   \end{align*}
   \]
What is Studied?

- The set of all clones of $P_k$ forms a lattice $\mathbb{L}(P_k)$ with respect to inclusion $\subseteq$.

- The lattice $\mathbb{L}(P_k)$ is the main object of study in clone theory.
  1. $\mathbb{L}(P_1)$ contains exactly one element.
  2. $\mathbb{L}(P_2)$ is countable.
  3. $\mathbb{L}(P_k) (k \geq 3)$ has the cardinality of the continuum.
Main Tools: Idea of a Function Preserving a Relation

Let $R_k$ be the set of all finitary predicates on $\{0, 1, \ldots, k - 1\}$.

**Definition.** $f^{(n)} \in P_k$ preserves $p^{(r)} \in R_k$ if

$$[a_1 \in p^{(r)}, \ldots, a_n \in p^{(r)}] \implies f^{(n)}(a_1, \ldots, a_n) \in p^{(r)}.$$
Rosenberg (1970)

Description of all maximal clones in $P_k$ based on the idea of a function preserving a relation.

**Theorem [Rosenberg]**

*Every maximal clone in $P_k$ has the form*

$$Pol \, \rho \equiv \{ f \in P_k \mid f \text{ preserves } \rho \}$$

*where $\rho \in R_k^{(r)}$ ($1 \leq r \leq k$).*
Main Tools: The Galois Connection Between Clones and Relational Clones

Krasner (1939, 1945, 1968/69)

The connection between algebras and relations was introduced. A special Galois connection between subsets of the permutation groups

\[ S_n \equiv \{ f \in P_k(1) \mid f \text{ is bijective} \} \]

and subsets of \( R_k \) was formulated.
Main Tools: The Galois Connection Between Clones and Relational Clones

Geiger (1968); Bodnarchuk et al. (1969)

Galois theory for function algebras and relation algebra.

**Theorem [Geiger; Bodnarchuk et al.]**

There exists a one-to-one correspondence between clones and relational clones.

- If $F \subseteq P_k$ is a clone, then

  $\text{Pol Inv } F = F$.

- If $M \subseteq R_k$ is a relational clone, then

  $\text{Inv Pol } M = M$. 

Main Tools: The Galois Connection Between Clones and Relational Clones

\[ \text{Inv}: \mathcal{F} \mapsto \{ p \in R_k \mid \forall f \in \mathcal{F}: f \text{ preserves } p \} \]

\[ \text{Pol}: \mathcal{M} \mapsto \{ f \in P_k \mid \forall p \in \mathcal{M}: f \text{ preserves } p \} \]
Solution: Weak Closure Operator

Remark (on consistency of weak closure operator)

Weak closure $[\ ]_{weak}$ is consistent with the closure under all primitive positive formulas:

$$[[M]_{weak}]_{p.p.f} = [M]_{p.p.}.$$
Some properties of the lattice of all relational clones of degree $\leq 2$.

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total number of classes</td>
<td>2,079,040</td>
</tr>
<tr>
<td>Number of maximal subclasses</td>
<td>18</td>
</tr>
<tr>
<td>The longest chain size</td>
<td>55</td>
</tr>
<tr>
<td>Number of maximal subclones of a given clone</td>
<td>0—30</td>
</tr>
<tr>
<td></td>
<td>(average $\approx 5.8$)</td>
</tr>
</tbody>
</table>