A problem on squares and its application to the square packing problem

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Introduction

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Given an $n \times n$ square $ABCD$, determine the minimum number of points, denoted by $\mathrm{punct}(n)$, that need to be chosen inside the square $ABCD$ such that there does not exists a unit square inside the square $ABCD$ containing none of the chosen points in its interior.

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- Applications to the (quite researched) square packing problem.

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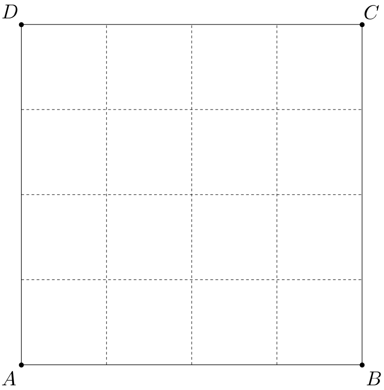

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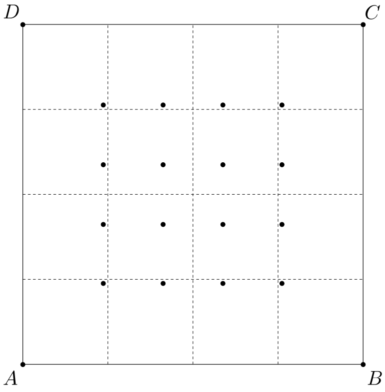

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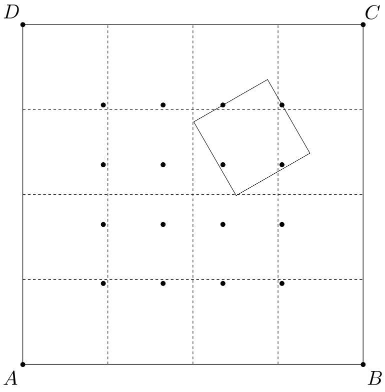

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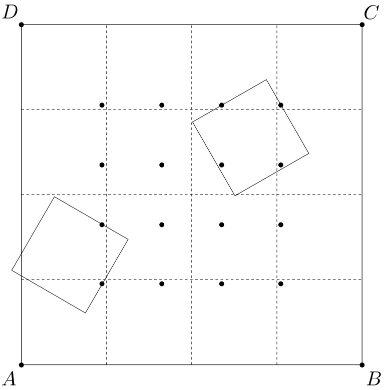

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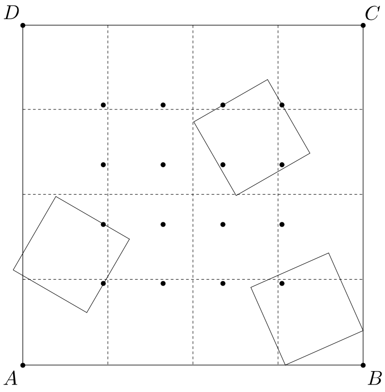

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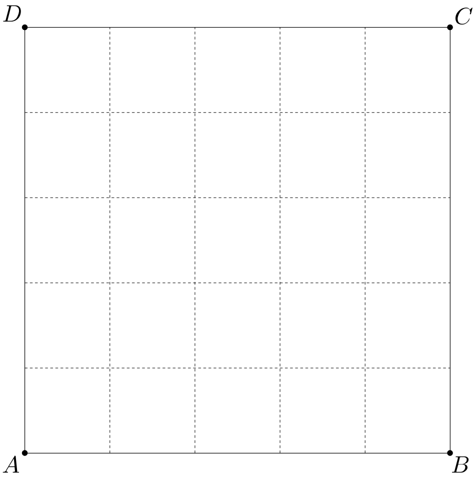

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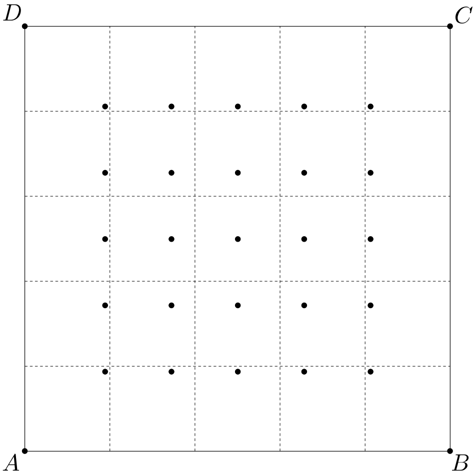

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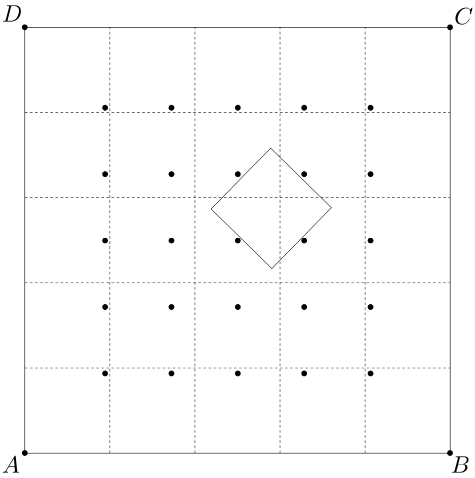

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upper bound	1	4	9	16	25	36	49	72	90	110
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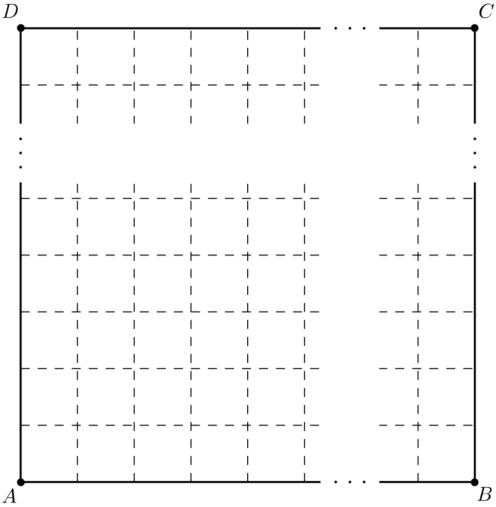

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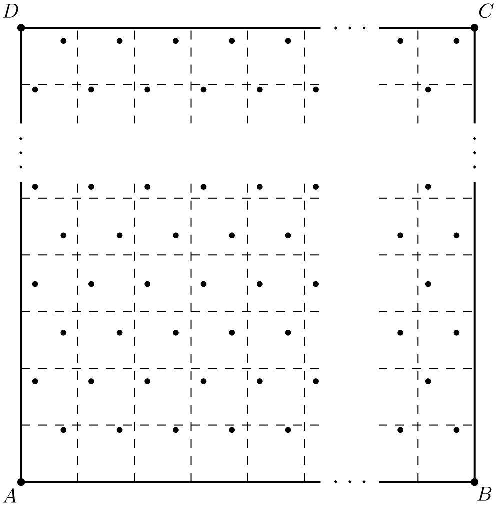

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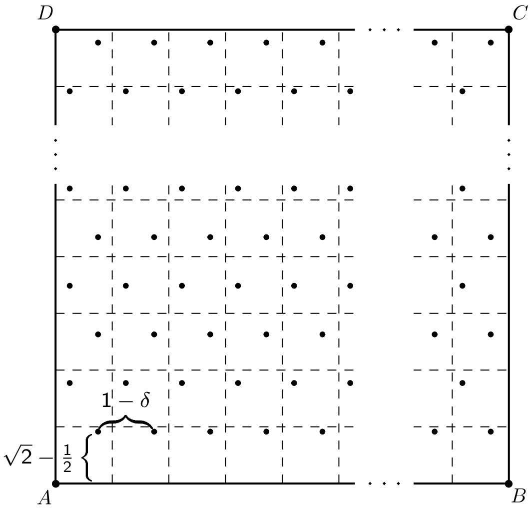

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Corollary

If $\min\{m,n\} \leqslant 7$, then

$$\mathrm{punct}(m,n) = mn.$$

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For any $x > 0$,

$$\operatorname{punct}(x) \leqslant \begin{cases} \lfloor x \rfloor \left(\left\lfloor \frac{2}{\sqrt{3}}(x + 1 - 2\sqrt{2}) \right\rfloor + 2 \right), & \text{if } \{x\} < \frac{1}{2}; \\ \end{cases}$$

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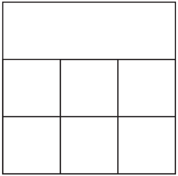

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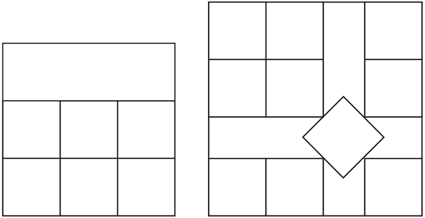

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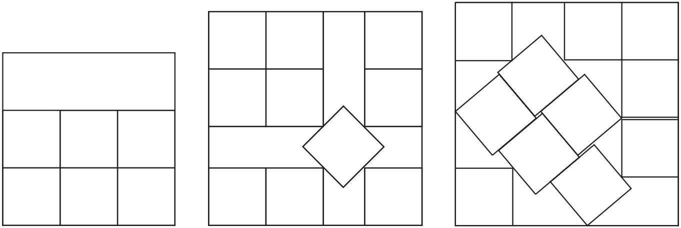

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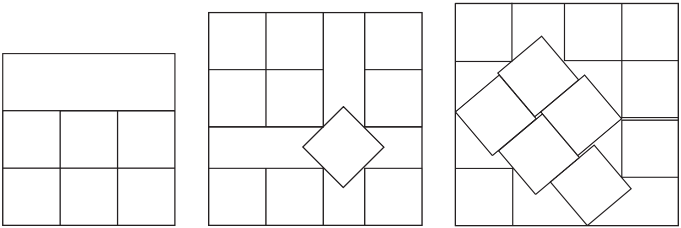

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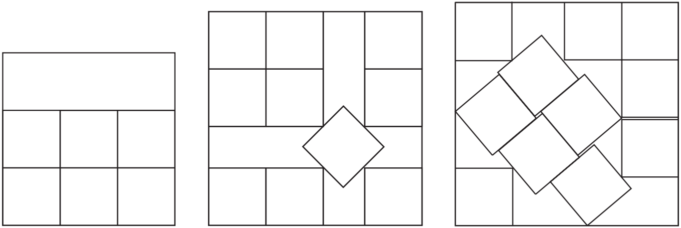

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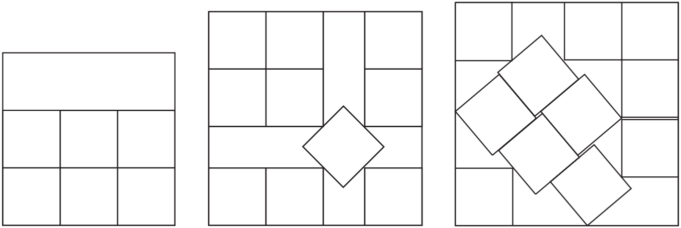

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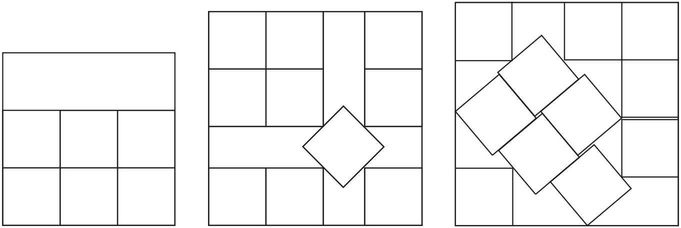

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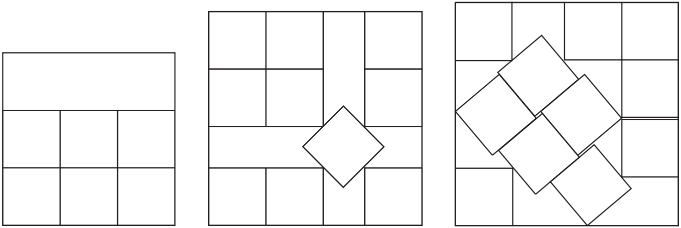

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- Trivia: for $n = 11$, it is known that the packing shown above is better than any packing where each square is rotated by 0° or 45°.

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Theorem

The values of $s(n)$ for some values of n are as given below:

n	$s(n)$	*originally proved by*	n	$s(n)$	*originally proved by*
8	3	*Bajmóczy, $<$'79*	35	6	*Friedman '98*
15	4	*El Moumni '99*	46	7	*Bentz '10*
23	5	*Nagamochi '05*	47	7	*Nagamochi '05*
24	5	*Friedman '98*	48	7	*Nagamochi '05*
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- $s(61) \geqslant s \left(\mathrm{punct} \left(\frac{7\sqrt{3}}{2} + 2\sqrt{2} - 1 - \varepsilon \right) + 1 \right) > \frac{7\sqrt{3}}{2} + 2\sqrt{2} - 1 - \varepsilon$

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- $\left\lfloor \frac{2}{\sqrt{3}} \left(\left(\frac{7\sqrt{3}}{2} + 2\sqrt{2} - 1 - \varepsilon \right) + 1 - 2\sqrt{2} \right) \right\rfloor = \left\lfloor \frac{2}{\sqrt{3}} \left(\frac{7\sqrt{3}}{2} - \varepsilon \right) \right\rfloor = 6$
- $\mathrm{punct}\left(\frac{7\sqrt{3}}{2} + 2\sqrt{2} - 1 - \varepsilon \right) \leqslant 7 \cdot (6+2) + \left\lfloor \frac{6+2}{2} \right\rfloor = 56 + 4 = 60$
- $s(61) \geqslant s\left(\mathrm{punct}\left(\frac{7\sqrt{3}}{2} + 2\sqrt{2} - 1 - \varepsilon \right) + 1 \right) > \frac{7\sqrt{3}}{2} + 2\sqrt{2} - 1 - \varepsilon$
- $s(61) \geqslant \frac{7\sqrt{3}}{2} + 2\sqrt{2} - 1$

■

A conjecture about asymptotical tightness

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Conjecture

For $n \longrightarrow \infty$, we have

$$\mathrm{punct}(n) \sim \frac{2}{\sqrt{3}} n^2.$$