Clonoids and Promise CSP

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- A more natural setting for the algebraic approach
- Many new computationally interesting questions: approximation problems
- A broader dichotomy conjecture
- The missing piece seems universal algebraic in nature

Promise constraint satisfaction

Fix a pair of finite relational structures $\mathbb{A}, \mathbb{B}$ such that $\mathbb{A} \to \mathbb{B}$

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- Related to approximation: given a satisfiable instance of a hard problem, find an approx. solution (weaker constraints)

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- Believed NP-hard if $c \geq 3$, open already for $c = 3, d = 5$

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Theorem (Austrin, Håstad, Guruswami FOCS'14)

$\mathrm{SAT}(\alpha)$ is tractable for $\alpha \geq \frac{1}{2}$ and NP-hard for $\alpha < \frac{1}{2}$

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Theorem (Brakensiek, Guruswami '16; for CSP: Jeavons '98)

If $\mathrm{Pol}(\mathbb{A}, \mathbb{B}) \subseteq \mathrm{Pol}(\mathbb{C}, \mathbb{D})$, then $\mathrm{PCSP}(\mathbb{C}, \mathbb{D}) \leq_{\mathrm{L}} \mathrm{PCSP}(\mathbb{A}, \mathbb{B})$.

Some results translate directly

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A boolean $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ is solvable by the Bulatov, Dalmau algorithm (based on compact representations) if and only if $x_1 + \dots + x_{2k+1}$ (mod 2) is a polymorphism for every $k > 0$

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- Quotients, subalgebras, products: all generalize naturally

Linear Birkhoff Theorem

Theorem (Barto, Opršal, Pinsker '15)

Let $\mathcal{C}, \mathcal{D}$ be clones. There exists a $\mathrm{h1}$ clone homomorphism $\Phi : \mathcal{C} \to \mathcal{D}$ if and only if $\mathcal{D} \in \mathsf{ERP}(\mathcal{C})$

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  E.g., if $f(x, x, y) = g(y, y, x, x)$ for all $x, y \in C$,
  then $\Phi(f)(x, x, y) = \Phi(g)(y, y, x, x)$ for all $x, y \in D$

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- E, R, $\mathsf{P}_{\mathsf{fin}}$ stronger than H, S, $\mathsf{P}_{\mathsf{fin}}$ but still provides a logspace reduction of the corresponding PCSPs

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- E, R, $\mathsf{P}_{\mathsf{fin}}$ stronger than H, S, $\mathsf{P}_{\mathsf{fin}}$ but still provides a logspace reduction of the corresponding PCSPs

Corollary (Opršal '17; for CSP: Barto, Opršal, Pinsker '15)

If there exists a clonoid homomorphism from $\mathsf{Pol}(\mathbb{A}, \mathbb{B})$ to $\mathsf{Pol}(\mathbb{C}, \mathbb{D})$, then $\mathrm{PCSP}(\mathbb{C}, \mathbb{D}) \leq_{\mathrm{L}} \mathrm{PCSP}(\mathbb{A}, \mathbb{B})$

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On the other hand:

> *"At an intuitive level, we might expect a* PCSP *to be easy if there are weak polymorphisms that 'genuinely' depend on a lot of variables, and hard if a few variables exert a lot of influence on the function. The precise way to formalize this notion that captures the boundary between tractable and hard is not yet clear."*

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- A similar condition holds for applicability of some algorithms

The mother of all inapproximability results

GapLabelCover(C, ϵ) — C colors, $\epsilon > 0$

- **Input:** A bipartite graph $G = \langle U \cup V; E \rangle$ and a set of constraint functions $\sigma_{uv} : C \to C$ for every edge $uv \in E$
- A coloring $\lambda : G \to C$ satisfies an edge if $\sigma_{uv}(\lambda(u)) = \lambda(v)$
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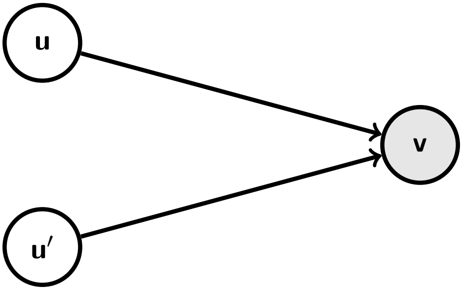
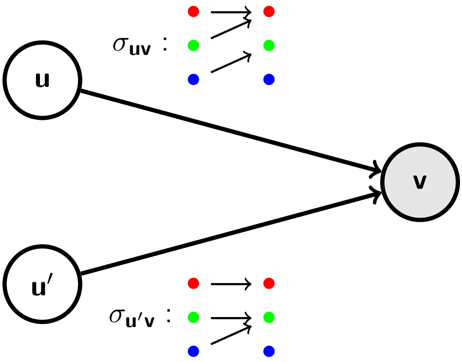
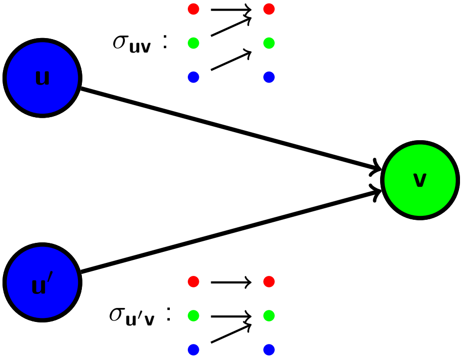
Theorem (Raz '95)

$\forall \epsilon > 0 \; \exists C$ GapLabelCover(C, ϵ) is NP-hard

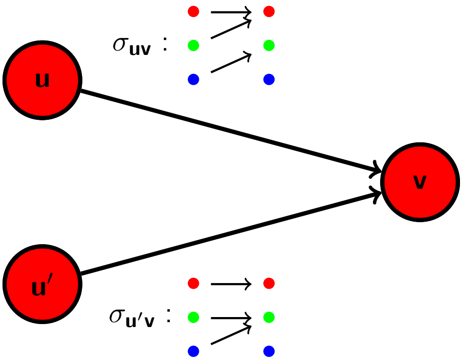

u
v
u′



σ_{uv} :
u
v
u′
$\sigma_{u'v}$:



$\sigma_{\mathbf{uv}}$:
u
v
u′
$\sigma_{\mathbf{u'v}}$: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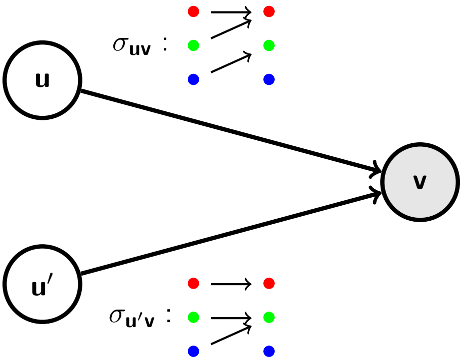

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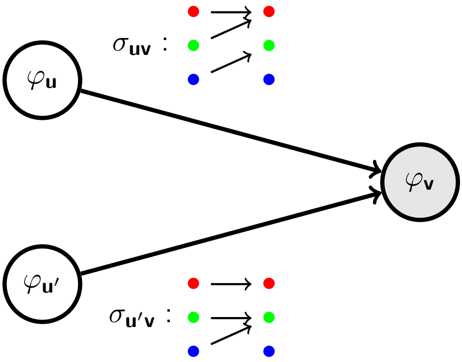

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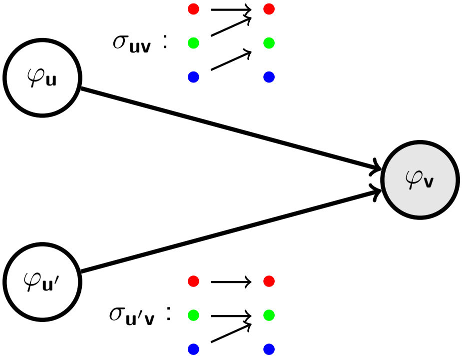

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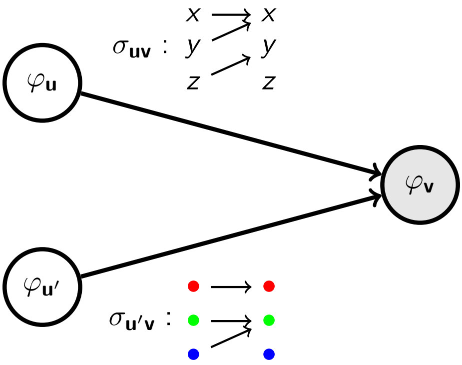

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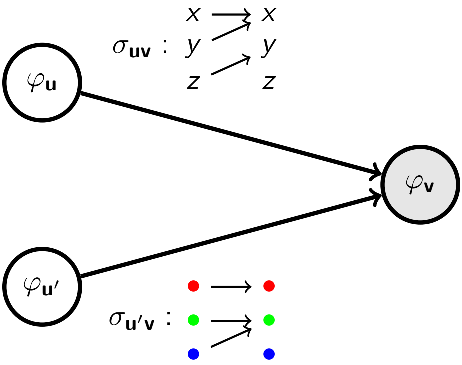

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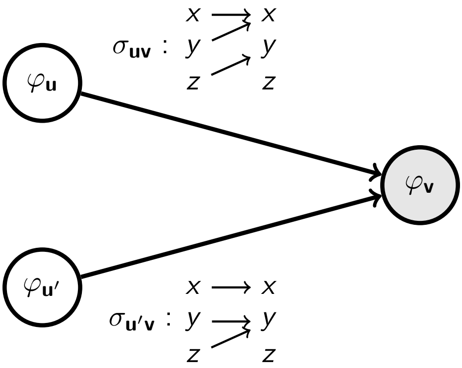

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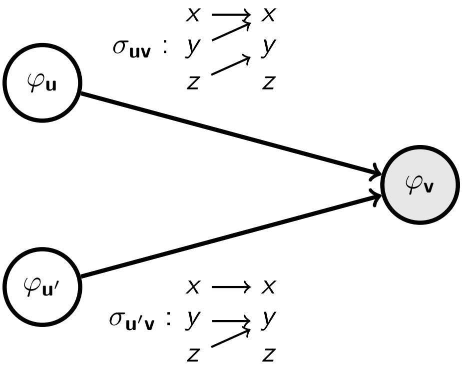

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$\Rightarrow$ the h1 identity $\varphi_u(x, x, y) = \varphi_{u'}(x, y, y)$

A universal hardness reduction?

- A reduction from $\mathsf{GapLabelCover}(C, \epsilon)$ to satisfiability of systems of $\mathrm{h1}$ identities in $\mathsf{Pol}_C(\mathbb{A}, \mathbb{B})$

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- A reduction from GapLabelCover(C, ϵ) to satisfiability of systems of $\mathrm{h1}$ identities in $\mathsf{Pol}_C(\mathbb{A}, \mathbb{B})$
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- A reduction from $\mathsf{GapLabelCover}(C, \epsilon)$ to satisfiability of systems of $\mathrm{h1}$ identities in $\mathsf{Pol}_C(\mathbb{A}, \mathbb{B})$
- **Completeness:** YES instance of LabelCover $\Rightarrow$ $\mathrm{h1}$ identities satisfiable by projections ("dictatorship test")
- **Soundness?** Intuition: "You cannot satisfy a large system of $\mathrm{h1}$ identities without either having a Taylor operation of large enough arity or using at least some 'projections'."