Coherency for monoids

AAA94 + NSAC 2017
Novi Sad, June 15-18th 2017
Special session dedicated to Prof. Siniša Crvenković

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Coherency for Monoids: a finitary condition

Throughout, S will denote a monoid.

Finitary condition

A condition satisfied by all finite monoids.

Example

Every right ideal of S is finitely generated, i.e. S is **weakly right Noetherian**.

Finitary conditions were introduced by **Noether** and **Artin** in the early 20th Century to study rings; they changed the course of algebra entirely.

Example

Every right congruence on S is finitely generated, i.e. S is **right Noetherian**.

Coherency for Monoids: the definition

Coherency

This is the finitary condition of importance to us today

Definition

S is right coherent if every finitely generated S-subact of every finitely presented right S-act is finitely presented.[a]

[a] This definition comes from Wheeler (1976)

Left coherency is defined dually: S is coherent if it is left and right coherent.

Acts over monoids: S-acts
Representation of monoid S by mappings of sets

A **(right)** S**-act** is a set A together with a map

$$A \times S \to A, \ (a, s) \mapsto as$$

such that for all $a \in A, s, t \in S$

$$a1 = a \text{ and } (as)t = a(st).$$

Beware: an S-act is also called an S-set, S-system, S-action, S-operand, or S-polygon.

Acts over monoids: S-acts
Standard definitions/Elementary observations

- S-acts form a variety of universal algebras, to which we may apply the usual notions of subalgebra (S-**subact**), congruence, morphism (S-**morphism**), factor/quotient S-act, finitely generated, etc.
- S-acts and S-morphisms form a category, **Act-S**.
- We have usual definitions of **free, projective, injective**, etc. including variations on **flat**.
- Free S-acts are **disjoint unions of copies of** S.

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- We have usual definitions of **free, projective, injective**, etc. including variations on **flat**.
- Free S-acts are **disjoint unions of copies of** S.
- A is **finitely presented** if

$$A \cong F_S(X)/\rho$$

for some finitely generated free S-act $F_S(X)$ and finitely generated congruence ρ.

Which monoids are right coherent?
First observations
Coherency is a finitary condition!

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Theorem: Normak (77)

If S is right noetherian then S is right coherent.

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If S is right noetherian then S is right coherent.

Example: Fountain (92)

There is a monoid S which is weakly right noetherian but which is not right coherent.

Why is coherency interesting?

Theorem: Wheeler (1976); G (19[redacted]), Ivanov (1992)

The following are equivalent for a monoid S:

1. S is right coherent;
2. the existentially closed S-acts form an axiomatisable class;
3. the first-order theory of S-acts has a model companion.

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- Certain nice classes of monoids are coherent (more later).
- It has connections with products and ultraproducts of flat left S-acts (**Bulman-Fleming and McDowell, G, Sedaghatjoo**).
- Coherency is related to **purity** (more later).

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20th century

Theorem: G (1992)

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- groups;
- semilattices;

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Theorem: G (1992)

The following monoids are right coherent:

- the free commutative monoid on X;
- Clifford monoids;
- groups;
- semilattices;
- regular monoids for which every right ideal is principal.

Which monoids are right coherent?
21st century

The free monoid X^* on X is given by

$$X^* = \{x_1x_2 \ldots x_n : n \geq 0, x_i \in X\}$$

with

$$(x_1x_2 \ldots x_n)(y_1y_2 \ldots y_m) = x_1x_2 \ldots x_ny_1x_2 \ldots y_m.$$

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Theorem: G, Hartmann, Ruškuc (2015)

Any free monoid X^* is coherent.

Theorem: K.G. Choo, K.Y. Lam and E. Luft (1972)

Free rings are coherent.

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21st century

Theorem: G, Hartmann (2016)

Free inverse monoids are NOT right coherent.
Free left restriction monoids are right coherent.

With several authors we have further results determining coherency for certain monoids; closure properties etc. Many questions remain!

Question

Is $\mathcal{I}_X$ coherent?

Why am I interested in coherency?
Absolutely pure and almost pure S-acts

Let A be an S-act. An **equation** over A has the form

$$xs = xt, \; xs = yt \text{ or } xs = a$$

where x, y are variables, $s, t \in S$ and $a \in A$.

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A set of equations is **consistent** if it has a solution in some S-act $B \supseteq A$.

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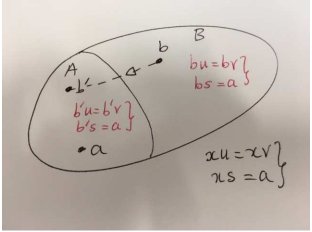
An S-act A is **absolutely pure (almost pure)** if every finite consistent set of equations over A (in 1 variable) has a solution in A.



Why am I interested in coherency? Absolute vs almost purity

Let $\mathcal{A}$ ($\mathcal{A}_1$) denote the class of absolutely pure (almost pure) S-acts.

Definition: A monoid is completely right pure

if every S-act is absolutely pure.

Theorem: G (19[redacted])

A monoid S is completely right pure if and only if all S-acts are almost pure.

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This enabled me to characterise **completely right pure monoids (1991)** in a way analogous to that of Skornjakov (1979), and Fountain (1974) and Isbell (1972) for **completely right injective monoids** .

A Question: We know $\mathcal{A}_1 \subseteq \mathcal{A}$

Does there exist a monoid S with $\mathcal{A} \neq \mathcal{A}_1$???

Theorem: G, Yang Dandan, Salma Shaheen (2016)

Let S be a finite monoid and let A be an almost pure S-act. Then A is absolutely pure, i.e. $\mathcal{A} = \mathcal{A}_1$.

Consequently: $\mathcal{A} = \mathcal{A}_1$ is a finitary condition.

Why am I interested in coherency?
Purity: Absolute purity vs almost purity
The Question: does $\mathcal{A} = \mathcal{A}_1$?

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Theorem: G, Yang Dandan (2016/7)

Let S be a right coherent monoid. Then $\mathcal{A} = \mathcal{A}_1$.

Questions, Questions...!

- Does there exist a monoid S with $\mathcal{A} \neq \mathcal{A}_1$? i.e. an S-act A such that A is almost pure but not absolutely pure?

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- Does there exist a monoid S with $\mathcal{A} \neq \mathcal{A}_1$? i.e. an S-act A such that A is almost pure but not absolutely pure?
- Does $\mathcal{A} = \mathcal{A}_1$ if and only if S is right coherent?

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- Does there exist a monoid S with $\mathcal{A} \neq \mathcal{A}_1$? i.e. an S-act A such that A is almost pure but not absolutely pure?
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- Is $\mathcal{I}_X$ coherent?

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- Does $\mathcal{A} = \mathcal{A}_1$ if and only if S is right coherent?
- Is $\mathcal{I}_X$ coherent?
- Is the free left adequate/Ehresmann monoid right coherent?

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- Does there exist a monoid S with $\mathcal{A} \neq \mathcal{A}_1$? i.e. an S-act A such that A is almost pure but not absolutely pure?
- Does $\mathcal{A} = \mathcal{A}_1$ if and only if S is right coherent?
- Is $\mathcal{I}_X$ coherent?
- Is the free left adequate/Ehresmann monoid right coherent?
- Determine exact connections of right coherency with products/ultraproducts of flat **left** S-acts.

Thank you for listening

Happy retirement to Siniša!

For an S-act A we can build canonical absolutely pure (almost pure) extensions $A(\aleph_0)$ $(A(1))$.

Proposition G: 2017

The following are equivalent for a monoid S:

1. every almost pure S-act is absolutely pure;
2. for every **finitely generated subact of every finitely presented S-act** A, we have $A(1)$ is a retract of $A(\aleph_0)$.

Why am I interested in coherency?
Purity: absolute purity vs almost purity

Theorem: G (19 ■)

The following are equivalent for a monoid S:

1. S is right coherent;
2. $\mathcal{A}$ is axiomatisable;
3. $\mathcal{A}_1$ is axiomatisable.

Why am I interested in coherency?
Purity: absolute purity vs almost purity

Purity properties may be reformulated as weak injectivity properties.
Injectivity may be reformulated as a stronger purity property.

Definition: A monoid is completely right injective (completely right pure)

if every S-act is injective (absolutely pure).

Fountain (1974), Isbell (1972) (following work of **Skornjakov (69)** and others)

Characterised completely right injective monoids in terms of right ideals and elements.

Completely right injective monoids

Theorem: Skornjakov (1969)

A monoid S is completely right injective if S has a left zero and S satisfies (*) for any right ideal I of S and right congruence ρ on S, there is an $s \in I$ such that for all $u, v \in S, w \in I$, $sw \, \rho \, w$ and if $u \, \rho \, v$ then $su \, \rho \, sv$.

Theorem: Fountain (1974)

A monoid S is completely right injective if and only if S has a zero, and each right ideal I has an idempotent generator e such that, for each pair of elements $a, b \in S \setminus I$, we have $a'ea \, \mathcal{R} \, b'eb$ for all $a' \in V(a), b' \in V(b)$ implies that $a'ea = b'eb$ for all $a' \in V(a), b' \in V(b)$.