Lattice representations with DCC posets

Gergő Gyenizse

University of Szeged

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Preorder representations

Definition

The preorders of a set X are its reflexive and transitive relations. They form a lattice with respect to inclusion denoted by $\mathrm{Pre}X$.

Definition

A representation of a lattice $\mathbf{L}$ with preorders of X is an injective homomorphism from $\mathbf{L}$ into $\mathrm{Pre}X$.
Such a representation is a **representation with equivalences/posets/DCC posets** if all the images are symmetric/antisymmetric/antisymmetric containing no infinite descending chain.

Representability theorems

Theorem (Whitman, 1946)

All lattices are representable with equivalences.

Theorem (Pudlák, Tůma, 1980)

All finite lattices are representable with equivalences on a finite set.

Theorem (Achein, 1972)

All lattices are representable with posets.

Theorem (Sivak, 1978)

Not all finite lattices are representable with posets on a finite set. (See the next frame for the characterization.)

Lower bounded lattices

Definition

Let a and b be distinct join irreducible elements of a lattice $\mathbf{L}$. We call a **join dependent** of b (noted by $a\,D\,b$), if there is a $c \in L$ so that $a \leq b \vee c$, but $a \not\leq d \vee c$ for any $d < b$.

Definition

A lattice is **(McKenzie-) lower bounded** if it is finitely generated, and there is no D-circle in it (D being viewed as a digraph on $J(L)$, the set of join irreducible elements of $\mathbf{L}$).

Theorem

The lattices representable with posets on a finite set are precisely the finite lower bounded lattices.

How to represent a finite lattice with DCC posets?

- suppose $h : \mathbf{L} \mapsto \mathrm{Pre}X$ is the eventual representation,
- for any $(x_1, x_2) \in X^2$ that is an edge of an image of $\mathbf{L}$ there is a smallest l so that $(x_1, x_2) \in h(l)$,
- suppose $l \leq l_1 \vee l_2$ is a minimal non-trivial join covering of l (so the inequality does not hold if we decrease either l_1 or l_2)
- we need a y between x_1 and x_2, such that $(x_1, y) \in h(l_1 \vee l_2)$, while $(y, h_2) \in h(l_1) \cup h(l_2)$,
- the crucial choice is whether (y, h_2) is in $h(l_1)$ or $h(l_2)$,
- if l and l_1 are in a D-circle, then this choice has to be made infinitely many times, and only finitely many times can we choose $h(l_2)$

Condition for finite lattices

Definition

For a lattice $\mathbf{L}$, $\mathcal{C}_{\mathbf{L}}$ denotes the set of minimal nontrivial join covers, i.e. the set

$$\{(l, l_1, l_2) \in L^3 : l \leq l_1 \vee l_2, \forall l_1', l_2' : (l \not\leq l_1' \vee l_2, l \not\leq l_1 \vee l_2')\}$$

.

Theorem

For a finite lattice $\mathbf{L}$*, TFAE:*

1. $\mathbf{L}$ *is representable with DCC posets,*
2. *there is a coloring* $s : \mathcal{C}_{\mathbf{L}} \mapsto \{1, 2\}$ *(that is symmetrical in the last two variables) such that the binary relation*

$$\{(l, l_i) : (l, l_1, l_2) \in \mathcal{C}_{\mathbf{L}}, s(l, l_1, l_2) = i\}$$

does not contain a circle.

Sufficient condition for arbitrary lattices

Definition

For a lattice $\mathbf{L}$, $\mathcal{B}_{\mathbf{L}}$ denotes the set of all nontrivial join covers, i.e. the set

$$\{(l, l_1, l_2) \in L^3 : l \leq l_1 \vee l_2,\ l \not\leq l_1, l_2\}$$

.

Theorem

A lattice $\mathbf{L}$ is representable with DCC-posets if there is a coloring $s : \mathcal{B}_{\mathbf{L}} \mapsto \{1, 2\}$ (that is symmetrical in the last two variables) such that for the binary relation

$$\delta := \{(l, l_i) : (l, l_1, l_2) \in \mathcal{B}_{\mathbf{L}},\ s(l, l_1, l_2) = i\},$$

$\delta \vee \leq_{\mathbf{L}}$ does not contain a (non-loop) circle.

Necessary condition for arbitrary lattices

Definition

An element of a lattice c is completely join irreducible if there is a largest element c^* among all elements of the lattice smaller than c.

Definition

Suppose that $\underline{a} := a_1 \, D \, a_2 \, D \ldots D \, a_k \, D \, a_1$ and $\underline{b} := b_1 \, D \, b_2 \, D \ldots D \, b_l \, D \, b_1$ are D-cycles of $\mathbf{L}$ consisting of completely join irreducible elements. We say that $\underline{a}$ depends on $\underline{b}$ if there exist indices i and j satisfying:

- $a_{j+1} \leq b_i \vee a_j$,
- $a_{j+1} \not\leq b_i \vee a_j^*$,
- $a_{j+1} \not\leq b_i^* \vee a_j$

Theorem

If the dependancy relation on the set of D-cylces of $\mathbf{L}$ *contains a circle, then* $\mathbf{L}$ *cannot be represented with DCC posets.*

Representable lattices

The set of lattices representable with DCC-posets:

- is closed to taking sublattices and direct products,
- contains all lower bounded lattices,
- is not contained in the class of join semidistributive lattices,
- does not contain M_3.

The problem of deciding whether a finite lattice is representable by DCC-posets is in NP.

Open questions

- Is the aforementioned membership problem NP-hard?
- Can the necessary condition be made sufficient, or the sufficient necessary?
- What about complete embeddings of complete lattices into $\mathrm{Pre}X$ so that all the images are DCC-posets?
- Is there a nontrivial lattice quasi-identity that all representable lattices satisfy?
- Are there other embeddability problems that have the same answer as this one (or, at least, the same finite lattices being embeddable)?

Thank you for your attention!