Infinite algebgras with few subpowers

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History

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- Bulatov, Dalmau: *A simple algorithm for Mal'tsev constraints*, 2006.

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- Idziak, Marković, McKenzie, Valeriote, Willard: *Tractability and learnability arising from algebras with few subpowers*, 2007.

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- Kearnes, Szendrei: *Clones of algebras with parallelogram terms*, 2012.
- Aichinger, Mayr, McKenzie: *On the number of finite algebraic structures*, 2012.

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$$s_{\mathbf{A}}(n) \leq 2^{p(n)}.$$

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- The polymorphism clone of $(\mathbb{Q}, <)$ has few subpowers. (Every relation is determined by its binary projections.)
- Any finite algebra with a Mal'cev term (edge term) has few subpowers, *but*
- a vector space over an infinite field does not!

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2. $\mathbf{Z}_2 = (\{0, 1\}, n)$ with $n(x, y, z) = x + y + z$, the minority.
3. Interpret both m and n as 4-ary edge terms, and consider the product $\mathbf{Q} \times \mathbf{Z}_2$.
4. Add many unary operations preserving $<$ and linear idenitities in copies of $\mathbb{Q}$ and $\mathbb{Z}_2$.

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A local edge term of **A** on a (finite) set $S \subseteq A$ is a term t_S that satisfy the edge identities for all tuples of elements of S.

We say that an algebra **A** has local edge terms if it has a local edge term for every finite $S \subseteq A$.

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Small weak orbit growth is necessary for having few subpowers.

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A representation is compact if it contains at most $\binom{n}{k}|A|^k + 2n|A|^2$ tuples.

Weakly compact representations

Given a weakly oligomorphic algebra $\mathbf{A}$ with local k-edge terms and a subpower $S \leq \mathbf{A}^n$. We say that a set $R \subseteq S$ is a quasi representation of S if

- for all $\mathbf{a} \in S$ and every k-element subset I of coordinates there is $\mathbf{r} \in R$ such that $\mathrm{proj}_I\, \mathbf{a} \sim \mathrm{proj}_I\, \mathbf{r}$; and
- for all $i \leq n$ and $\mathbf{a}, \mathbf{b} \in S$ such that $a_j = b_j$ for all $j < i$ there are $\mathbf{c}, \mathbf{d} \in R$ such that $c_j = d_j$ for all $j < i$ and $(a_i, b_i) \sim (c_i, d_i)$.

A quasi representation is weakly compact if it contains at most $\binom{n}{k} o(k) + 2n\, o(2)$ tuples where $o(k)$ denote the number of weak orbits of k-tuples.

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