Infinitely many reducts of homogeneous structures

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TU Dresden

Novi Sad, 17th June 2017

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Two reduct are called interdefinable iff they are reducts of one another.

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To classify all reducts of a structure $\mathfrak{A}$ (up to interdefinability).

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- random ordered graph (Bodirsky, Pinsker, Pongrácz, 2014)

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Conjecture (Thomas, 1991)

Every homogeneous, finite relational structure has finitely many reducts.

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$n > \max(\mathrm{ar}(R_i))$, $a_1, a_2, \dots, a_n$ linearly independent.
Then $a_i \mapsto a_i : i < n$, $a_n \mapsto a_1 + a_2 + \cdots + a_{n-1}$ does not extend to an automorphism.

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3. the countably infinite dimensional affine space

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4. the countably infinite set with a constant 0

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In fact: there exists an infinite ascending chain of reducts.

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- find all the reducts of $(\mathcal{V}, c)$
- modify construction to disprove Thomas' Conjecture

Thank you for your attention!