Identities in plactic and related monoids

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(Joint work with various subsets of
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Young tableaux & Schensted's algorithm

Let $\mathcal{A} = \{1 < 2 < 3 < \ldots\}$ and let $\mathcal{A}_n = \{1 < 2 < 3 < \ldots < n\}$.

1	1	3	4	4
2	3	6		
5				

- Rows weakly increasing left to right.
- Columns strictly increasing top to bottom.
- Longer columns to the left.

Schensted's algorithm To insert $a \in \mathcal{A}$ into a tableau:

1. If adding a to the end of the top row gives a tableau, this is the result.
2. Otherwise, let b the leftmost symbol of the top row such that $b > a$. Replace b with a ('bumping b').
3. Recursively insert b into the tableau formed by all lower rows.

For a word $u = u_1 u_2 \cdots u_k \in \mathcal{A}^*$:

- Start with an empty tableau and insert u_1, then u_2, ..., finally u_k.
- Call the resulting tableau $\mathrm{P}_{\mathrm{plac}}(u)$. For example, $\mathrm{P}_{\mathrm{plac}}(2531613443)$ is the tableau above.

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The plactic monoid

Schensted's algorithm computes a tableau $\mathsf{P}_{\mathsf{plac}}(u)$ for $u \in \mathcal{A}^*$.

Define $u \equiv_{\mathsf{plac}} v \iff \mathsf{P}_{\mathsf{plac}}(u) = \mathsf{P}_{\mathsf{plac}}(v)$.

Theorem (Knuth 1970)

The relation $\equiv_{\mathsf{plac}}$ is a congruence on $\mathcal{A}^*$.

- $\mathsf{plac} = \mathcal{A}^*/\equiv_{\mathsf{plac}}$ is the plactic monoid.
- $\mathsf{plac}_n = \mathcal{A}_n^*/\equiv_{\mathsf{plac}}$ is the plactic monoid of rank n.

Clearly,

$$\mathsf{plac}_1 \hookrightarrow \mathsf{plac}_2 \hookrightarrow \ldots \hookrightarrow plac_n \hookrightarrow \mathsf{plac}_{n+1} \hookrightarrow \ldots \hookrightarrow \mathsf{plac}.$$

and

$$\mathsf{plac} = \bigcup_{n \in \mathbb{N}} \mathsf{plac}_n.$$

Identities

- An identity is a formal equality $u = v$, where $u, v \in X^*$.
- A monoid M satisfies $u = v$ if substituting any element of M for each symbol in X gives an equality that holds in M.

For example,

- Any commutative monoid satisfies $xy = yx$.
- Any nilpotent group of class 2 satisfies $xyzyx = yxzxy$ [Neumann & Taylor 1963].

An identity is trivial if u and v are the same word; otherwise non-trivial.

- $xy = xy$ is trivial.
- $xy = yx$ and $xyzyx = yxzxy$ are non-trivial.

Questions

- Does plac satisfy a non-trivial identity?
- Does each plac_n satisfy a non-trivial identity?

Chinese monoid

- The Chinese monoid is also defined by an insertion algorithm.
- The Chinese monoid is related to the plactic monoid by its growth type.
- plac_2 is isomorphic to the Chinese monoid of rank 2.

Proposition (Jaszuńska & Okniński)

The Chinese monoid embeds into a direct product of copies of the bicyclic monoid and the infinite cyclic group.

Proposition (Adian)

The bicyclic monoid satisfies $xyyxxyxyyx = xyyxyxxyyx$ ('Adian's identity').

Corollary

The Chinese monoid satisfies Adian's identity.

Corollary

plac_2 satisfies Adian's identity.

Identities for plac_3

Adian's identity $xyyxxyxyyx = xyyxyxxyyx$

Proposition (Kubat & Okniński)

plac_3 satisfies $pqqpqp = pqpqqp$, where p and q are the left and right sides of Adian's identity. plac_3 does not satisfy Adian's identity.

- Use detailed calculations using normal forms.

Proposition (Izhakian)

plac_3 satisfies $xyxy^2x^2yxyxyxy^2x^2y = xyxy^2x^2yyxxyxy^2x^2y$.

- Use a (complicated) representation in the monoid of 3×3 upper-triangular tropical matrices.

Proposition (Cain, Klein, Kubat, M., Okniński)

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Identities for plac and plac_n

Theorem (Cain, Klein, Kubat, M., Okniński)

plac does not satisfy any non-trivial identity.

Proposition (CKKMO)

plac_n does not satisfy any non-trivial identity of length less than or equal to n.

Theorem (Schensted 1961)

$$\text{Number of columns in } P_{\text{plac}}(u) = \begin{cases} \text{Length of the longest weakly} \\ \text{increasing subsequence of } u; \end{cases}$$

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Proof that plac_n satisfies no identity of length at most n

Suppose plac_n satisfies $u(x, y) = v(x, y)$ of length n.

Assume

$$u = u_1 \cdots u_{j-1} x u_{j+1} \cdots u_n$$
$$v = v_1 \cdots v_{j-1} y v_{j+1} \cdots v_n$$

Let $s = 12 \cdots n \in \mathcal{A}_n^*$,
$t = 12 \cdots (n - j)(n - j + 2) \cdots n \in \mathcal{A}_n^*$ (miss out $n - j + 1$).

So the tableaux $P_{\text{plac}}\big(u(s, t)\big)$ and $P_{\text{plac}}\big(v(s, t)\big)$ are equal.

Longest decreasing subsequences:

In $u(s, t)$: $u_1 \quad u_2 \quad \cdots \quad u_{j-1} \quad x \quad u_{j+1} \quad \cdots \quad u_{n-1} \quad u_n$

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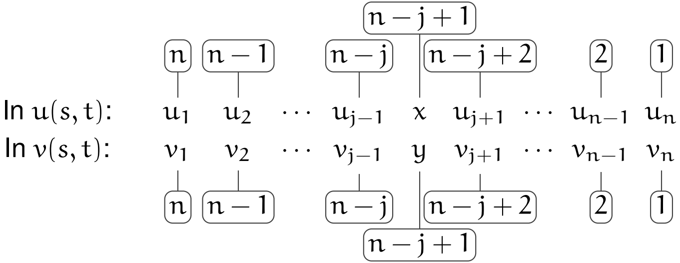

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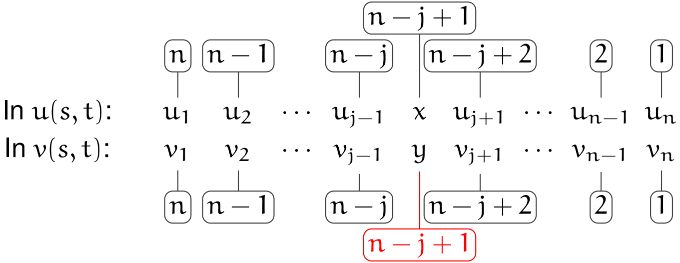
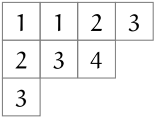


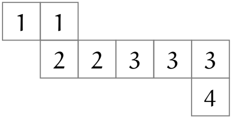
‘Plactic-like’ monoids

A family of monoids whose elements can be viewed as combinatorial objects:

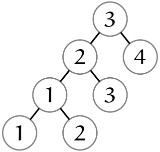
Plactic monoid
Young tableaux



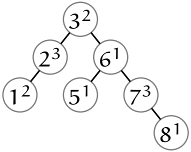
Hypoplactic monoid
Quasi-ribbon tableaux



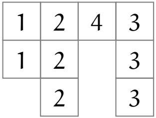
Sylvester monoid
Binary search trees



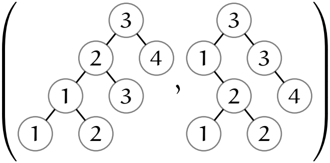
Taiga monoid
BSTs with multiplicities



Stalactite monoid
Stalactite tableaux

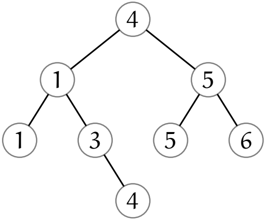


Baxter monoid
Pairs of twin binary search trees



Binary search trees and leaf insertion

Binary search tree (BST):



To insert a into a BST T:

- Add a as a leaf node in the unique position that yields a BST.

For a word $u = u_k u_{k-1} \cdots u_1 \in \mathcal{A}^*$.

- Start with an empty BST and insert u_1, then u_2, …, finally u_k.
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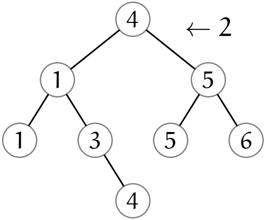

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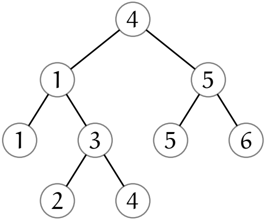

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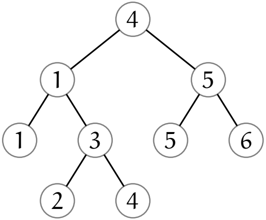

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Theorem (Hivert et al. 2005)

The relation $\equiv_{\mathrm{sylv}}$ is a congruence on $\mathcal{A}^*$.

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- $\mathrm{sylv}_n = \mathcal{A}_n^*/{\equiv_{\mathrm{sylv}}}$ is the sylvester monoid of rank n.

Theorem (Cain, M.)

sylv satisfies the identity $xyxy = yxxy$.
This is the unique shortest non-trivial identity satisfied by sylv.

- 'unique' up to renaming variables and swapping the two sides.

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- Want to show that $\mathsf{P}_{\mathsf{sylv}}(stst) = \mathsf{P}_{\mathsf{sylv}}(tsst)$ for all $s, t \in \mathcal{A}^*$.
- Suffices to prove that $\mathsf{P}_{\mathsf{sylv}}(pr) = \mathsf{P}_{\mathsf{sylv}}(qr)$ for all $p, q, r \in \mathcal{A}^*$, where p, q, r contain the same number of each symbol.

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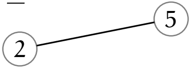

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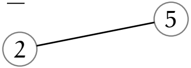

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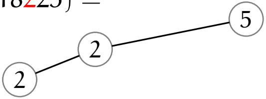

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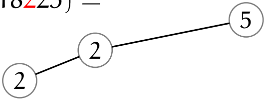

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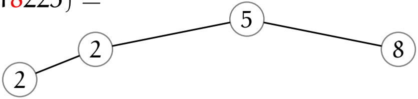

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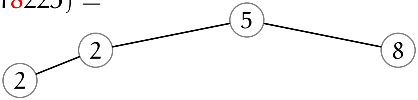

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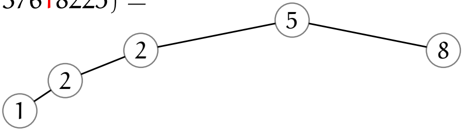

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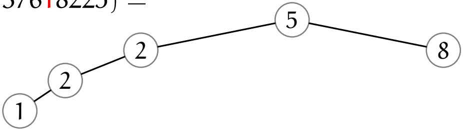

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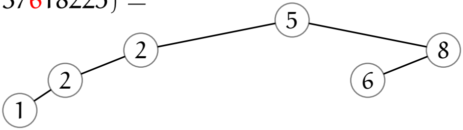

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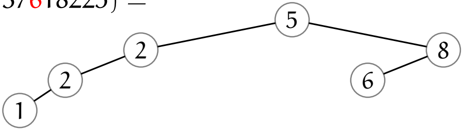

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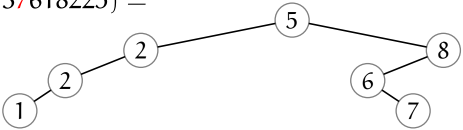

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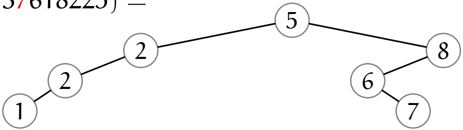

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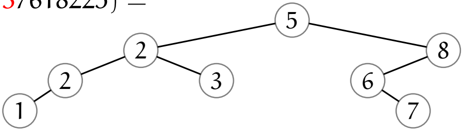

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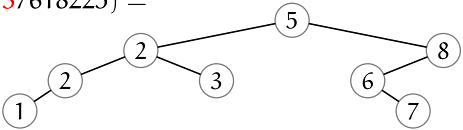

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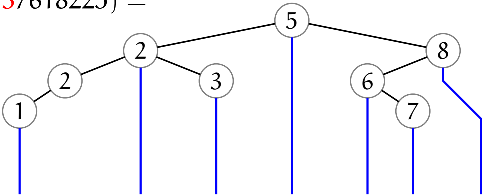

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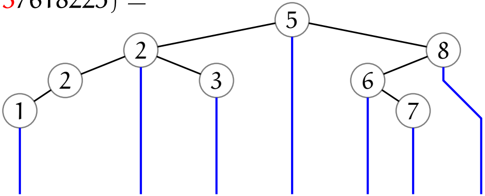

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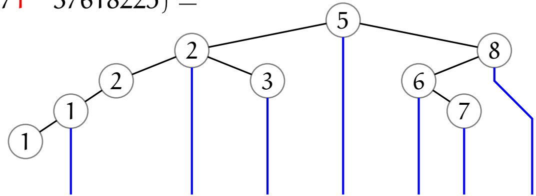

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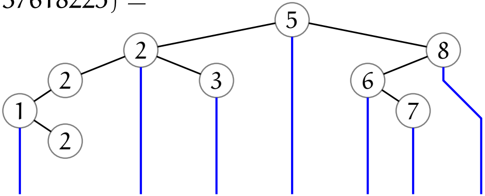

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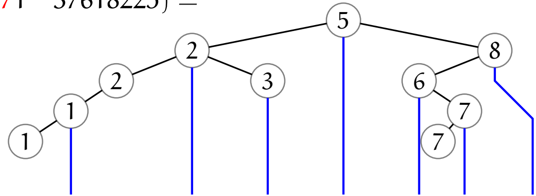

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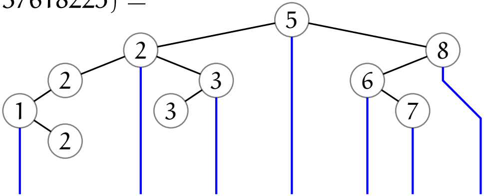

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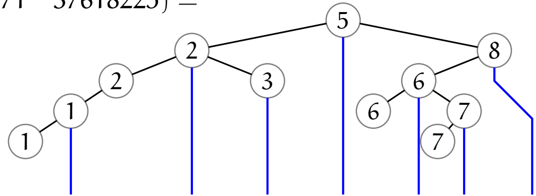

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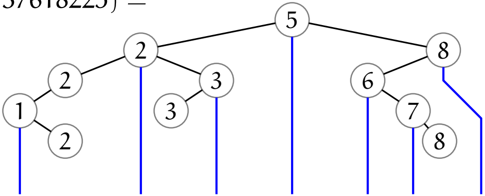

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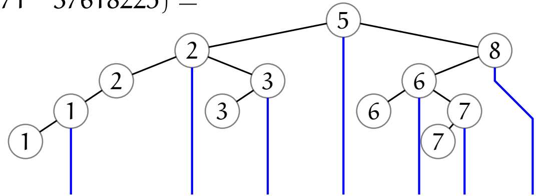

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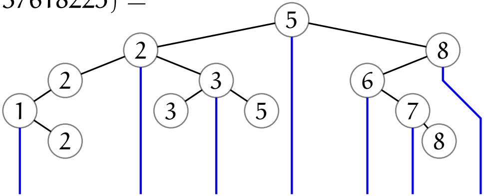

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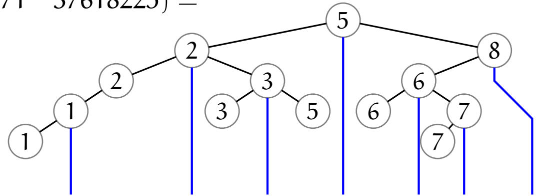

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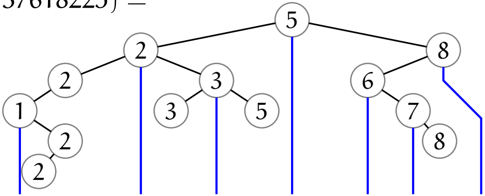

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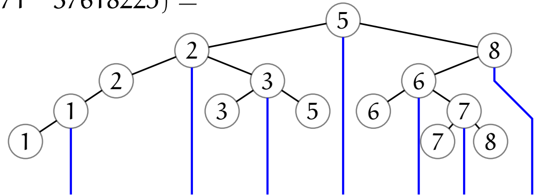

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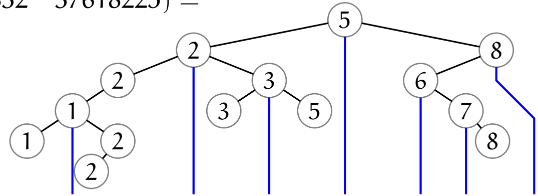

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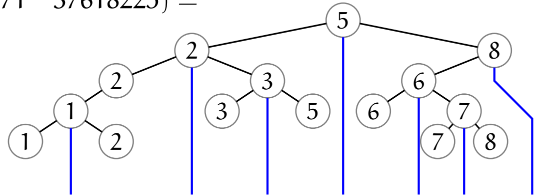

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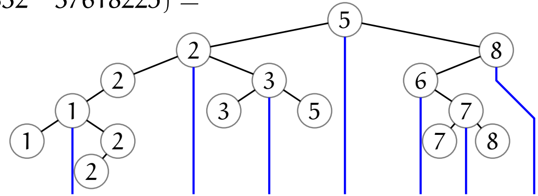

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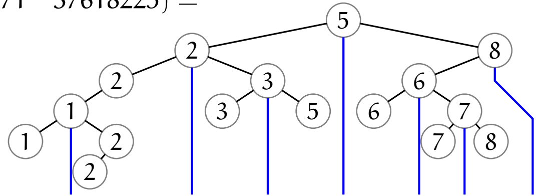

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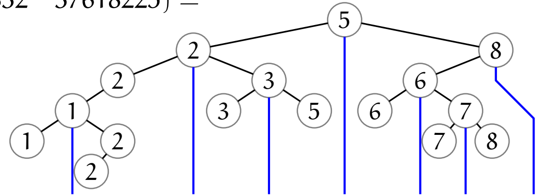

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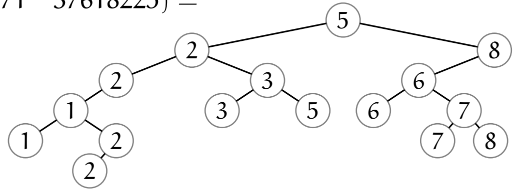

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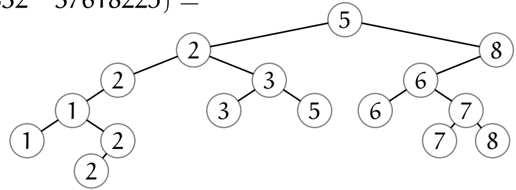

Quasi-ribbon tableaux & insertion

Quasi-ribbon tableau (QRT):

1	2			
	3	4	4	
			5	7

To insert a symbol a into a quasi-ribbon tableau T:

- Break the tableau two parts: $T_{\leqslant}$ is up to and including the bottom-right-most symbol r such that $r \leqslant a$; the remainder is $T_{>}$.
- Add a to the right of r.
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For a word $u = u_1 u_2 \cdots u_k \in \mathcal{A}^*$.

- Start with an empty QRT and insert u_1, then u_2, …, finally u_k.
- Call the resulting QRT $\mathrm{P}_{\mathsf{hypo}}(u)$. For example, $\mathrm{P}_{\mathsf{hypo}}(15344723)$is the QRT above.

Lemma

If $i < j$ are symbols in u and there is no k in u with $i < k < j$, then:

i and j are on the different rows of $\mathrm{P}_{\mathsf{hypo}}(u)$ $\iff$ In u, some symbol i is to the right of some symbol j

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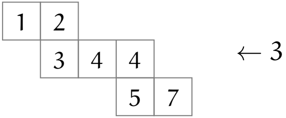

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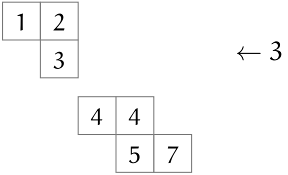

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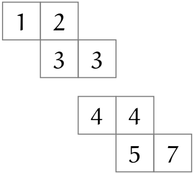

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- Start with an empty QRT and insert u_1, then u_2, ..., finally u_k.
- Call the resulting QRT $\mathrm{P}_{\mathrm{hypo}}(u)$. For example, $\mathrm{P}_{\mathrm{hypo}}(15344723)$is the QRT above.

Lemma

If $i < j$ are symbols in u and there is no k in u with $i < k < j$, then:

i and j are on the different rows of $\mathrm{P}_{\mathrm{hypo}}(u)$ $\Longleftrightarrow$ In u, some symbol i is to the right of some symbol j

Quasi-ribbon tableaux & insertion

Quasi-ribbon tableau (QRT):

1	2			
	3	3		
		4	4	
			5	7

To insert a symbol a into a quasi-ribbon tableau T:

- Break the tableau two parts: $T_{\leqslant}$ is up to and including the bottom-right-most symbol r such that $r \leqslant a$; the remainder is $T_{>}$.
- Add a to the right of r.
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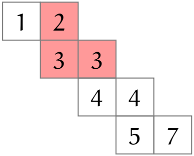

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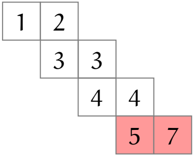

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The hypoplactic monoid

Define $u \equiv_{\text{hypo}} v \iff \mathsf{P}_{\text{hypo}}(u) = \mathsf{P}_{\text{hypo}}(v)$.

Theorem (Novelli 2000)

The relation $\equiv_{\text{hypo}}$ is a congruence on A^*.

- hypo $= \mathcal{A}^*/\equiv_{\text{hypo}}$ is the hypoplactic monoid.
- $\text{hypo}_n = \mathcal{A}_n^*/\equiv_{\text{hypo}}$ is the hypoplactic monoid of rank n.

Theorem (Cain, M.)

hypo satisfies the identities

$$xyxy = xyyx = yxxy = yxyx;$$
$$xxyx = xyxx.$$

These are the unique shortest non-trivial identities satisfied by hypo.

- A QRT is determined by the number of each symbol it contains and which symbols are on the same rows.
- These are the length-4 identities where the two sides preserve these properties.

Summary table

Monoid	*Symbol*	*Identity*	*In rank n*
Plactic	plac	None	?
Hypoplactic	hypo	$xyxy = yxyx$	Y
Sylvester	sylv	$xyxy = yxxy$	Y
Baxter	baxt	$yxxyxy = yxyxxy$	Y
Stalactic	stal	$xyx = yxx$	Y
Taiga	taig	$xyx = yxx$	Y
Left patience sorting	lPS	None	N
Right patience sorting	rPS	None	Y

Question

Does plac_n satisfy a non-trivial identity for $n \geqslant 4$?

- Conjectured hierarchy of identities for plac_n, length $2 \times 5^{n-1}$.
- Lots of random examples checked in plac_4 using Sage.