An algorithmic ordering condition for groups

George Metcalfe

Mathematical Institute
University of Bern

Joint work with Almudena Colacito

AAA 94, Novi Sad, 17. June 2017

A Key Theorem

Ordered groups play a significant role in *algebra*, *logic*, and *topology*.

The following fundamental result for the theory of ordered groups was proved by Shimbireva (1947), Iwasawa (1948), Neumann (1948), and Vinogradov (1949), after unpublished claims by Birkhoff and Tarski.

Theorem

Every free group can be totally ordered.

A Key Theorem

Ordered groups play a significant role in *algebra*, *logic*, and *topology*.

The following fundamental result for the theory of ordered groups was proved by Shimbireva (1947), Iwasawa (1948), Neumann (1948), and Vinogradov (1949), after unpublished claims by Birkhoff and Tarski.

Theorem

Every free group can be totally ordered.

This Talk

We give a proof of this theorem that

- uses a new **algorithmic** ordering condition for groups;
- relates ordering free groups to **validity** in totally ordered groups.

Ordered Groups and Proof Theory.
A. Colacito and G. Metcalfe.
Proceedings of WoLLIC 2017, to appear.

Proof Theory for Lattice-Ordered Groups.
N. Galatos and G. Metcalfe.
Annals of Pure and Applied Logic 167 (2016), 707–724.

This Talk

We give a proof of this theorem that

- uses a new **algorithmic** ordering condition for groups;
- relates ordering free groups to **validity** in totally ordered groups.

Ordered Groups and Proof Theory.
A. Colacito and G. Metcalfe.
Proceedings of WoLLIC 2017, to appear.

Proof Theory for Lattice-Ordered Groups.
N. Galatos and G. Metcalfe.
Annals of Pure and Applied Logic 167 (2016), 707–724.

This Talk

We give a proof of this theorem that

- uses a new **algorithmic** ordering condition for groups;
- relates ordering free groups to **validity** in totally ordered groups.

Ordered Groups and Proof Theory.
A. Colacito and G. Metcalfe.
Proceedings of WoLLIC 2017, to appear.

Proof Theory for Lattice-Ordered Groups.
N. Galatos and G. Metcalfe.
Annals of Pure and Applied Logic 167 (2016), 707–724.

This Talk

We give a proof of this theorem that

- uses a new **algorithmic** ordering condition for groups;
- relates ordering free groups to **validity** in totally ordered groups.

Ordered Groups and Proof Theory.
A. Colacito and G. Metcalfe.
Proceedings of WoLLIC 2017, to appear.

Proof Theory for Lattice-Ordered Groups.
N. Galatos and G. Metcalfe.
Annals of Pure and Applied Logic 167 (2016), 707–724.

Ordered Groups

A **partial order** of a group $\mathbf{G} = \langle G, \cdot, ^{-1}, e \rangle$ is a partial order $\leq$ of G that is preserved by multiplication on both sides, i.e., for all $a, b, c \in G$,

$$a \leq b \quad \Longrightarrow \quad ac \leq bc \quad \text{and} \quad ca \leq cb.$$

If $\leq$ is also total, then it is called a **total order** of $\mathbf{G}$.

Positive Cones

The **positive cone** $P_{\leq} = \{a \in G : e < a\}$ of a partial order $\leq$ of **G** satisfies

(1) $a, b \in P_{\leq} \implies ab \in P_{\leq}$

(2) $a \in P_{\leq}, b \in G \implies bab^{-1} \in P_{\leq}$

(3) $e \notin P_{\leq}$.

Conversely, if P is a normal subsemigroup of **G** omitting e, then

$$a \leq_P b \iff ba^{-1} \in P \cup \{e\}$$

defines a partial order of **G** such that $P = P_{\leq_P}$.

Positive Cones

The **positive cone** $P_{\leq} = \{a \in G : e < a\}$ of a partial order $\leq$ of **G** satisfies

(1) $a, b \in P_{\leq} \implies ab \in P_{\leq}$

(2) $a \in P_{\leq}, b \in G \implies bab^{-1} \in P_{\leq}$

(3) $e \notin P_{\leq}$.

Conversely, if P is a normal subsemigroup of **G** omitting e, then

$$a \leq_P b \iff ba^{-1} \in P \cup \{e\}$$

defines a partial order of **G** such that $P = P_{\leq_P}$.

Positive Cones

The **positive cone** $P_{\leq} = \{a \in G : e < a\}$ of a partial order $\leq$ of **G** satisfies

(1) $a, b \in P_{\leq} \implies ab \in P_{\leq}$

(2) $a \in P_{\leq},\ b \in G \implies bab^{-1} \in P_{\leq}$

(3) $e \notin P_{\leq}$.

Conversely, if P is a normal subsemigroup of **G** omitting e, then

$$a \leq_P b \iff ba^{-1} \in P \cup \{e\}$$

defines a partial order of **G** such that $P = P_{\leq_P}$.

Positive Cones

The **positive cone** $P_{\leq} = \{a \in G : e < a\}$ of a partial order $\leq$ of **G** satisfies

(1) $a, b \in P_{\leq} \implies ab \in P_{\leq}$

(2) $a \in P_{\leq},\ b \in G \implies bab^{-1} \in P_{\leq}$

(3) $e \notin P_{\leq}$.

Conversely, if P is a normal subsemigroup of **G** omitting e, then

$$a \leq_P b \iff ba^{-1} \in P \cup \{e\}$$

defines a partial order of **G** such that $P = P_{\leq_P}$.

Positive Cones

The **positive cone** $P_{\leq} = \{a \in G : e < a\}$ of a partial order $\leq$ of **G** satisfies

(1) $a, b \in P_{\leq} \implies ab \in P_{\leq}$

(2) $a \in P_{\leq},\, b \in G \implies bab^{-1} \in P_{\leq}$

(3) $e \notin P_{\leq}$.

Conversely, if P is a normal subsemigroup of **G** omitting e, then

$$a \leq_P b \iff ba^{-1} \in P \cup \{e\}$$

defines a partial order of **G** such that $P = P_{\leq_P}$.

Positive Cones

The **positive cone** $P_{\leq} = \{a \in G : e < a\}$ of a partial order $\leq$ of **G** satisfies

(1) $a, b \in P_{\leq} \implies ab \in P_{\leq}$

(2) $a \in P_{\leq},\ b \in G \implies bab^{-1} \in P_{\leq}$

(3) $e \notin P_{\leq}$.

Conversely, if P is a normal subsemigroup of **G** omitting e, then

$$a \leq_P b \iff ba^{-1} \in P \cup \{e\}$$

defines a partial order of **G** such that $P = P_{\leq_P}$.

Fuchs' Orderability Condition

Let $\langle\langle S \rangle\rangle$ denote the normal subsemigroup generated by $S \subseteq G$; this is a partial order of **G** if and only if it doesn't contain e.

Theorem (Fuchs 1963)

The following are equivalent for any $S \subseteq G$:

(1) *S extends to a total order of G.*

(2) *For all $c_1, \ldots, c_n \in G \setminus \{e\}$, there exist $\delta_1, \ldots, \delta_n \in \{-1, 1\}$ such that*

$$e \notin \langle\langle S \cup \{c_1^{\delta_1}, \ldots, c_n^{\delta_n}\} \rangle\rangle.$$

Fuchs' Orderability Condition

Let $\langle\langle S \rangle\rangle$ denote the normal subsemigroup generated by $S \subseteq G$; this is a partial order of **G** if and only if it doesn't contain e.

Theorem (Fuchs 1963)

The following are equivalent for any $S \subseteq G$:

(1) *S extends to a total order of G.*

(2) *For all $c_1, \ldots, c_n \in G \setminus \{e\}$, there exist $\delta_1, \ldots, \delta_n \in \{-1, 1\}$ such that*

$$e \notin \langle\langle S \cup \{c_1^{\delta_1}, \ldots, c_n^{\delta_n}\} \rangle\rangle.$$

An Algorithmic Condition

We write $\vdash_{\mathbf{G}} S$ if a subset $S \subseteq G$ is derivable using the rules

$$\frac{}{T \cup \{a, a^{-1}\}} \qquad \frac{T \cup \{a\} \quad T \cup \{b\}}{T \cup \{ab\}} \qquad \frac{T \cup \{ba\}}{T \cup \{ab\}}$$

Example

In the group $\mathbf{Z} = \langle \mathbb{Z}, +, -, 0 \rangle$, we obtain $\vdash_{\mathbf{Z}} \{3, -5\}$ as follows:

$$\dfrac{\dfrac{}{\{3,-3\}} \qquad \dfrac{\dfrac{}{\{2,-2\}} \qquad \dfrac{}{\{1,-2\}}}{\{3,-2\}}}{\{3,-5\}}$$

An Algorithmic Condition

We write $\vdash_{\mathbf{G}} S$ if a subset $S \subseteq G$ is derivable using the rules

$$\frac{}{T \cup \{a, a^{-1}\}} \qquad \frac{T \cup \{a\} \quad T \cup \{b\}}{T \cup \{ab\}} \qquad \frac{T \cup \{ba\}}{T \cup \{ab\}}$$

Example

In the group $\mathbf{Z} = \langle \mathbb{Z}, +, -, 0 \rangle$, we obtain $\vdash_{\mathbf{Z}} \{3, -5\}$ as follows:

$$\dfrac{\dfrac{}{\{3,-3\}} \qquad \dfrac{\dfrac{}{\{2,-2\}} \qquad \dfrac{\dfrac{}{\{1,-1\}} \quad \dfrac{}{\{1,-1\}}}{\{1,-2\}}}{\{3,-2\}}}{\{3,-5\}}$$

An Algorithmic Condition

We write $\vdash_{\mathbf{G}} S$ if a subset $S \subseteq G$ is derivable using the rules

$$\frac{}{T \cup \{a, a^{-1}\}} \qquad \frac{T \cup \{a\} \quad T \cup \{b\}}{T \cup \{ab\}} \qquad \frac{T \cup \{ba\}}{T \cup \{ab\}}$$

Example

In the group $\mathbf{Z} = \langle \mathbb{Z}, +, -, 0 \rangle$, we obtain $\vdash_{\mathbf{Z}} \{3, -5\}$ as follows:

$$\frac{\dfrac{}{\{3,-3\}} \qquad \dfrac{\dfrac{}{\{2,-2\}} \qquad \dfrac{\dfrac{}{\{1,-1\}} \quad \dfrac{}{\{1,-1\}}}{\{1,-2\}}}{\{3,-2\}}}{\{3,-5\}}$$

An Algorithmic Condition

We write $\vdash_{\mathbf{G}} S$ if a subset $S \subseteq G$ is derivable using the rules

$$\frac{}{T \cup \{a, a^{-1}\}} \qquad \frac{T \cup \{a\} \quad T \cup \{b\}}{T \cup \{ab\}} \qquad \frac{T \cup \{ba\}}{T \cup \{ab\}}$$

Example

In the group $\mathbf{Z} = \langle \mathbb{Z}, +, -, 0 \rangle$, we obtain $\vdash_{\mathbf{Z}} \{3, -5\}$ as follows:

$$\dfrac{\dfrac{}{\{3,-3\}} \qquad \dfrac{\dfrac{}{\{2,-2\}} \qquad \dfrac{\dfrac{}{\{1,-1\}} \quad \dfrac{}{\{1,-1\}}}{\{1,-2\}}}{\{3,-2\}}}{\{3,-5\}}$$

An Algorithmic Condition

We write $\vdash_{\mathbf{G}} S$ if a subset $S \subseteq G$ is derivable using the rules

$$\frac{}{T \cup \{a, a^{-1}\}} \qquad \frac{T \cup \{a\} \quad T \cup \{b\}}{T \cup \{ab\}} \qquad \frac{T \cup \{ba\}}{T \cup \{ab\}}$$

Example

In the group $\mathbf{Z} = \langle \mathbb{Z}, +, -, 0 \rangle$, we obtain $\vdash_{\mathbf{Z}} \{3, -5\}$ as follows:

$$\dfrac{\dfrac{}{\{3,-3\}} \quad \dfrac{\dfrac{}{\{2,-2\}} \quad \dfrac{\dfrac{}{\{1,-1\}} \quad \dfrac{}{\{1,-1\}}}{\{1,-2\}}}{\{3,-2\}}}{\{3,-5\}}$$

A New Ordering Condition

Theorem

The following are equivalent for any $S \subseteq G$*:*

(1) S *does not extend to a total order of* $\mathbf{G}$.

(2) *There exist* $c_1, \ldots, c_n \in G \setminus \{e\}$ *such that for all* $\delta_1, \ldots, \delta_n \in \{-1, 1\}$,

$$e \in \langle\langle S \cup \{c_1^{\delta_1}, \ldots, c_n^{\delta_n}\}\rangle\rangle.$$

(3) $\vdash_{\mathbf{G}} S$.

Note that (1) $\Leftrightarrow$ (2) is just Fuchs' condition written contrapositively.

$(3) \Rightarrow (2)$

We prove by induction on the height of a $\vdash_{\mathbf{G}}$-derivation of S that

$$\vdash_{\mathbf{G}} S \quad \Longrightarrow \quad \begin{array}{l} \text{there exist } c_1, \ldots, c_n \in G \setminus \{e\} \text{ such that} \\ e \in \langle\langle S \cup \{c_1^{\delta_1}, \ldots, c_n^{\delta_n}\} \rangle\rangle \text{ for all } \delta_1, \ldots, \delta_n \in \{-1, 1\}. \end{array}$$

$(2) \Rightarrow (3)$

To prove:

$$\left.\begin{array}{l}\text{There exist } c_1, \ldots, c_n \in G \setminus \{e\} \text{ such that} \\ e \in \langle\langle S \cup \{c_1^{\delta_1}, \ldots, c_n^{\delta_n}\}\rangle\rangle \text{ for all } \delta_1, \ldots, \delta_n \in \{-1, 1\}\end{array}\right. \implies \vdash_{\mathbf{G}} S.$$

Crucial lemma:

For all $S_1 \cup S_2 \subseteq G$ and $c \in G \setminus \{e\}$,

$$\vdash_{\mathbf{G}} S_1 \cup \{c\} \quad \text{and} \quad \vdash_{\mathbf{G}} S_2 \cup \{c^{-1}\} \quad \implies \quad \vdash_{\mathbf{G}} S_1 \cup S_2.$$

$(2) \Rightarrow (3)$

To prove:

$$\left.\begin{array}{l}\text{There exist } c_1, \ldots, c_n \in G \setminus \{e\} \text{ such that} \\ e \in \langle\langle S \cup \{c_1^{\delta_1}, \ldots, c_n^{\delta_n}\} \rangle\rangle \text{ for all } \delta_1, \ldots, \delta_n \in \{-1, 1\}\end{array}\right. \quad \Longrightarrow \quad \vdash_{\mathbf{G}} S.$$

Crucial lemma:

For all $S_1 \cup S_2 \subseteq G$ and $c \in G \setminus \{e\}$,

$$\vdash_{\mathbf{G}} S_1 \cup \{c\} \ \text{ and } \ \vdash_{\mathbf{G}} S_2 \cup \{c^{-1}\} \quad \Longrightarrow \quad \vdash_{\mathbf{G}} S_1 \cup S_2.$$

A Restricted System

For any $S \cup \{c\} \subseteq G$, we call $\langle c, S \rangle$ a **pointed subset** of G, and write $\vdash^r_{\mathbf{G}} \langle c, S \rangle$ if it is derivable using the rules

$$\frac{}{\langle c, T \cup \{a, a^{-1}\} \rangle} \qquad \frac{}{\langle c, T \cup \{c^{-1}\} \rangle}$$

$$\frac{\langle c, T \cup \{a\} \rangle \quad \langle c, T \cup \{b\} \rangle}{\langle c, T \cup \{ab\} \rangle} \qquad \frac{\langle c, T \cup \{ba\} \rangle}{\langle c, T \cup \{ab\} \rangle}$$

We are then able to prove that

$$\vdash_{\mathbf{G}} S \cup \{c\} \iff \vdash^r_{\mathbf{G}} \langle c, S \rangle$$

and establish the crucial lemma by reasoning about $\vdash^r_{\mathbf{G}}$-derivations.

A Restricted System

For any $S \cup \{c\} \subseteq G$, we call $\langle c, S\rangle$ a **pointed subset** of G, and write $\vdash^r_{\mathbf{G}} \langle c, S\rangle$ if it is derivable using the rules

$$\frac{}{\langle c, T \cup \{a, a^{-1}\}\rangle} \qquad \frac{}{\langle c, T \cup \{c^{-1}\}\rangle}$$

$$\frac{\langle c, T \cup \{a\}\rangle \quad \langle c, T \cup \{b\}\rangle}{\langle c, T \cup \{ab\}\rangle} \qquad \frac{\langle c, T \cup \{ba\}\rangle}{\langle c, T \cup \{ab\}\rangle}$$

We are then able to prove that

$$\vdash_{\mathbf{G}} S \cup \{c\} \iff \vdash^r_{\mathbf{G}} \langle c, S\rangle$$

and establish the crucial lemma by reasoning about $\vdash^r_{\mathbf{G}}$-derivations.

A Restricted System

For any $S \cup \{c\} \subseteq G$, we call $\langle c, S \rangle$ a **pointed subset** of G, and write $\vdash^r_{\mathbf{G}} \langle c, S \rangle$ if it is derivable using the rules

$$\frac{}{\langle c, T \cup \{a, a^{-1}\}\rangle} \qquad \frac{}{\langle c, T \cup \{c^{-1}\}\rangle}$$

$$\frac{\langle c, T \cup \{a\}\rangle \quad \langle c, T \cup \{b\}\rangle}{\langle c, T \cup \{ab\}\rangle} \qquad \frac{\langle c, T \cup \{ba\}\rangle}{\langle c, T \cup \{ab\}\rangle}$$

We are then able to prove that

$$\vdash_{\mathbf{G}} S \cup \{c\} \iff \vdash^r_{\mathbf{G}} \langle c, S \rangle$$

and establish the crucial lemma by reasoning about $\vdash^r_{\mathbf{G}}$-derivations.

Free Groups and Totally Ordered Groups

Let **F** be a free group, and let $\mathcal{OG}$ be the class of **totally ordered groups** in an algebraic signature with operation symbols $\wedge$, $\vee$, $\cdot$, $^{-1}$, and e.

Theorem

The following are equivalent for $t_1, \ldots, t_m \in F$:

(1) $\{t_1, \ldots, t_m\}$ *does not extend to a total order of* **F**.

(2) *There exist $s_1, \ldots, s_n \in F \backslash \{e\}$ such that for all $\delta_1, \ldots, \delta_n \in \{-1, 1\}$,*

$$e \in \langle\langle \{t_1, \ldots, t_m, s_1^{\delta_1}, \ldots, s_n^{\delta_n}\} \rangle\rangle.$$

(3) $\vdash_{\mathbf{F}} \{t_1, \ldots, t_m\}$.

(4) $\mathcal{OG} \models e \leq t_1 \vee \ldots \vee t_m$.

Free Groups and Totally Ordered Groups

Let **F** be a free group, and let $\mathcal{OG}$ be the class of **totally ordered groups** in an algebraic signature with operation symbols $\wedge$, $\vee$, $\cdot$, ${}^{-1}$, and e.

Theorem

The following are equivalent for $t_1, \ldots, t_m \in F$:

(1) $\{t_1, \ldots, t_m\}$ *does not extend to a total order of* **F**.

(2) *There exist $s_1, \ldots, s_n \in F \backslash \{e\}$ such that for all $\delta_1, \ldots, \delta_n \in \{-1, 1\}$,*

$$e \in \langle\langle\{t_1, \ldots, t_m, s_1^{\delta_1}, \ldots, s_n^{\delta_n}\}\rangle\rangle.$$

(3) $\vdash_{\mathbf{F}} \{t_1, \ldots, t_m\}$.

(4) $\mathcal{OG} \models e \leq t_1 \vee \ldots \vee t_m$.

Free Groups and Totally Ordered Groups

Let **F** be a free group, and let $\mathcal{OG}$ be the class of **totally ordered groups** in an algebraic signature with operation symbols $\wedge$, $\vee$, $\cdot$, ${}^{-1}$, and e.

Theorem

The following are equivalent for $t_1, \ldots, t_m \in F$:

(1) $\{t_1, \ldots, t_m\}$ *does not extend to a total order of* **F**.

(2) *There exist $s_1, \ldots, s_n \in F \backslash \{e\}$ such that for all $\delta_1, \ldots, \delta_n \in \{-1, 1\}$,*

$$e \in \langle\langle \{t_1, \ldots, t_m, s_1^{\delta_1}, \ldots, s_n^{\delta_n}\} \rangle\rangle.$$

(3) $\vdash_{\mathbf{F}} \{t_1, \ldots, t_m\}$.

(4) $\mathcal{OG} \models e \leq t_1 \vee \ldots \vee t_m$.

An Example

We can show that $\vdash_{\mathbf{F}} \{xx, yy, x^{-1}y^{-1}\}$

$$\dfrac{\dfrac{\{x, yy, x^{-1}\} \quad \{x, yy, x^{-1}\}}{\{xx, yy, x^{-1}\}} \qquad \dfrac{}{\{xx, yy, y^{-1}\}}}{\{xx, yy, x^{-1}y^{-1}\}}$$

and hence, by our main theorem,

1. $\mathcal{OG} \models e \leq xx \vee yy \vee x^{-1}y^{-1}$
2. $\{xx, yy, x^{-1}y^{-1}\}$ does not extend to a total order of F.

An Example

We can show that $\vdash_{\mathbf{F}} \{xx, yy, x^{-1}y^{-1}\}$

$$\dfrac{\dfrac{\dfrac{}{\{x, yy, x^{-1}\}} \quad \dfrac{}{\{x, yy, x^{-1}\}}}{\{xx, yy, x^{-1}\}} \qquad \dfrac{\{xx, y, y^{-1}\} \quad \{xx, y, y^{-1}\}}{\{xx, yy, y^{-1}\}}}{\{xx, yy, x^{-1}y^{-1}\}}$$

and hence, by our main theorem,

1. $\mathcal{OG} \models e \leq xx \vee yy \vee x^{-1}y^{-1}$
2. $\{xx, yy, x^{-1}y^{-1}\}$ does not extend to a total order of $\mathbf{F}$.

An Example

We can show that $\vdash_{\mathbf{F}} \{xx, yy, x^{-1}y^{-1}\}$

$$\dfrac{\dfrac{\overline{\{x, yy, x^{-1}\}} \quad \overline{\{x, yy, x^{-1}\}}}{\{xx, yy, x^{-1}\}} \qquad \dfrac{\overline{\{xx, y, y^{-1}\}} \quad \overline{\{xx, y, y^{-1}\}}}{\{xx, yy, y^{-1}\}}}{\{xx, yy, x^{-1}y^{-1}\}}$$

and hence, by our main theorem,

1. $\mathcal{OG} \models e \leq xx \vee yy \vee x^{-1}y^{-1}$
2. $\{xx, yy, x^{-1}y^{-1}\}$ does not extend to a total order of $\mathbf{F}$.

An Example

We can show that $\vdash_{\mathbf{F}} \{xx, yy, x^{-1}y^{-1}\}$

$$\dfrac{\dfrac{\overline{\{x, yy, x^{-1}\}} \quad \overline{\{x, yy, x^{-1}\}}}{\{xx, yy, x^{-1}\}} \qquad \dfrac{\overline{\{xx, y, y^{-1}\}} \quad \overline{\{xx, y, y^{-1}\}}}{\{xx, yy, y^{-1}\}}}{\{xx, yy, x^{-1}y^{-1}\}}$$

and hence, by our main theorem,

1. $\mathcal{OG} \models e \leq xx \vee yy \vee x^{-1}y^{-1}$
2. $\{xx, yy, x^{-1}y^{-1}\}$ does not extend to a total order of F.

An Example

We can show that $\vdash_{\mathbf{F}} \{xx, yy, x^{-1}y^{-1}\}$

$$\dfrac{\dfrac{\overline{\{x, yy, x^{-1}\}} \quad \overline{\{x, yy, x^{-1}\}}}{\{xx, yy, x^{-1}\}} \quad \dfrac{\overline{\{xx, y, y^{-1}\}} \quad \overline{\{xx, y, y^{-1}\}}}{\{xx, yy, y^{-1}\}}}{\{xx, yy, x^{-1}y^{-1}\}}$$

and hence, by our main theorem,

1. $\mathcal{OG} \models e \leq xx \vee yy \vee x^{-1}y^{-1}$
2. $\{xx, yy, x^{-1}y^{-1}\}$ does not extend to a total order of $\mathbf{F}$.

An Example

We can show that $\vdash_{\mathbf{F}} \{xx, yy, x^{-1}y^{-1}\}$

$$\dfrac{\dfrac{\overline{\{x, yy, x^{-1}\}} \quad \overline{\{x, yy, x^{-1}\}}}{\{xx, yy, x^{-1}\}} \quad \dfrac{\overline{\{xx, y, y^{-1}\}} \quad \overline{\{xx, y, y^{-1}\}}}{\{xx, yy, y^{-1}\}}}{\{xx, yy, x^{-1}y^{-1}\}}$$

and hence, by our main theorem,

1. $\mathcal{OG} \models e \leq xx \vee yy \vee x^{-1}y^{-1}$
2. $\{xx, yy, x^{-1}y^{-1}\}$ does not extend to a total order of $\mathbf{F}$.

Since $\mathcal{OG} \not\models e \leq x$ for any variable x, by the previous theorem...

Corollary

Every free group can be totally ordered.

Ordering Free Groups

Since $\mathcal{OG} \not\models e \leq x$ for any variable x, by the previous theorem...

Corollary

Every free group can be totally ordered.

A More General Version

Let $\mathcal{V}$ be a variety of groups and $\mathbf{F}_\mathcal{V}$ a non-trivial free group of $\mathcal{V}$, and let $\mathcal{K}_\mathcal{V}$ be the class of totally ordered groups with group reducts in $\mathcal{V}$.

Theorem

The following are equivalent for $t_1, \ldots, t_m \in F_\mathcal{V}$:

(1) $\{t_1, \ldots, t_m\}$ *does not extend to a total order of* $\mathbf{F}_\mathcal{V}$.

(2) *There exist* $s_1, \ldots, s_n \in F_\mathcal{V} \setminus \{e\}$ *such that for all* $\delta_1, \ldots, \delta_n \in \{-1, 1\}$,

$$e \in \langle\langle \{t_1, \ldots, t_m, s_1^{\delta_1}, \ldots, s_n^{\delta_n}\} \rangle\rangle.$$

(3) $\vdash_{\mathbf{F}_\mathcal{V}} \{t_1, \ldots, t_m\}$.

(4) $\mathcal{K}_\mathcal{V} \models e \leq t_1 \vee \ldots \vee t_m$.

Corollary

$\mathbf{F}_\mathcal{V}$ *can be totally ordered if and only if* $\mathcal{K}_\mathcal{V}$ *is non-trivial.*

A More General Version

Let $\mathcal{V}$ be a variety of groups and $\mathbf{F}_\mathcal{V}$ a non-trivial free group of $\mathcal{V}$, and let $\mathcal{K}_\mathcal{V}$ be the class of totally ordered groups with group reducts in $\mathcal{V}$.

Theorem

The following are equivalent for $t_1, \ldots, t_m \in F_\mathcal{V}$:

(1) $\{t_1, \ldots, t_m\}$ *does not extend to a total order of* $\mathbf{F}_\mathcal{V}$.

(2) *There exist* $s_1, \ldots, s_n \in F_\mathcal{V} \backslash \{e\}$ *such that for all* $\delta_1, \ldots, \delta_n \in \{-1, 1\}$,

$$e \in \langle\langle \{t_1, \ldots, t_m, s_1^{\delta_1}, \ldots, s_n^{\delta_n}\} \rangle\rangle.$$

(3) $\vdash_{\mathbf{F}_\mathcal{V}} \{t_1, \ldots, t_m\}$.

(4) $\mathcal{K}_\mathcal{V} \models e \leq t_1 \vee \ldots \vee t_m$.

Corollary

$\mathbf{F}_\mathcal{V}$ *can be totally ordered if and only if* $\mathcal{K}_\mathcal{V}$ *is non-trivial.*

A More General Version

Let $\mathcal{V}$ be a variety of groups and $\mathbf{F}_{\mathcal{V}}$ a non-trivial free group of $\mathcal{V}$, and let $\mathcal{K}_{\mathcal{V}}$ be the class of totally ordered groups with group reducts in $\mathcal{V}$.

Theorem

The following are equivalent for $t_1, \ldots, t_m \in F_{\mathcal{V}}$:

(1) $\{t_1, \ldots, t_m\}$ *does not extend to a total order of* $\mathbf{F}_{\mathcal{V}}$.

(2) *There exist* $s_1, \ldots, s_n \in F_{\mathcal{V}} \backslash \{e\}$ *such that for all* $\delta_1, \ldots, \delta_n \in \{-1, 1\}$,

$$e \in \langle\langle\{t_1, \ldots, t_m, s_1^{\delta_1}, \ldots, s_n^{\delta_n}\}\rangle\rangle.$$

(3) $\vdash_{\mathbf{F}_{\mathcal{V}}} \{t_1, \ldots, t_m\}$.

(4) $\mathcal{K}_{\mathcal{V}} \models e \leq t_1 \vee \ldots \vee t_m$.

Corollary

$\mathbf{F}_{\mathcal{V}}$ *can be totally ordered if and only if* $\mathcal{K}_{\mathcal{V}}$ *is non-trivial.*

Outlook

- We have used new algorithmic condition for ordering groups to relate ordering relatively free groups to validity in classes of ordered groups.
- Can we use our conditions to prove ordering results for other groups (e.g., braid groups, fundamental groups of surfaces)?
- Can we use ordering theorems to develop a uniform proof theory for varieties of lattice-ordered groups?
- Is checking validity in totally ordered groups – equivalently, extending finite subsets of free groups to total orders – decidable?

Outlook

- We have used new algorithmic condition for ordering groups to relate ordering relatively free groups to validity in classes of ordered groups.
- Can we use our conditions to prove ordering results for other groups (e.g., braid groups, fundamental groups of surfaces)?
- Can we use ordering theorems to develop a uniform proof theory for varieties of lattice-ordered groups?
- Is checking validity in totally ordered groups – equivalently, extending finite subsets of free groups to total orders – decidable?

Outlook

- We have used new algorithmic condition for ordering groups to relate ordering relatively free groups to validity in classes of ordered groups.
- Can we use our conditions to prove ordering results for other groups (e.g., braid groups, fundamental groups of surfaces)?
- Can we use ordering theorems to develop a uniform proof theory for varieties of lattice-ordered groups?
- Is checking validity in totally ordered groups – equivalently, extending finite subsets of free groups to total orders – decidable?

Outlook

- We have used new algorithmic condition for ordering groups to relate ordering relatively free groups to validity in classes of ordered groups.
- Can we use our conditions to prove ordering results for other groups (e.g., braid groups, fundamental groups of surfaces)?
- Can we use ordering theorems to develop a uniform proof theory for varieties of lattice-ordered groups?
- Is checking validity in totally ordered groups – equivalently, extending finite subsets of free groups to total orders – decidable?