Representation of integral quantales by tolerances

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Definition 1. *A quantale is an algebraic structure* $\mathbf{Q} = (Q, \vee, \cdot)$ *such that* $(Q, \leq)$ *is a complete lattice (induced by the join operation* $\vee$*) and* $(Q, \cdot)$ *is a semigroup satisfying*

$$a \cdot \left(\bigvee\{b_i \mid i \in I\}\right) = \bigvee\{ab_i \mid i \in I\}, \tag{1}$$

$$\left(\bigvee\{b_i \mid i \in I\}\right) \cdot a = \bigvee\{b_i a \mid i \in I\} \tag{2}$$

for all $a \in Q$ *and* $b_i \in Q$, $i \in I$.

Definition 2. *A quantale* $\mathbf{Q}$ *is called:*

- **commutative** *if its multiplicative semigroup is commutative;*
- **unital** *if its multiplicative semigroup is a monoid;*
- **integral** *if the greatest element* 1 *of the semilattice* $(Q, \vee)$ *is the multiplicative identity element of* $(Q, \cdot)$.
- **involutive** *if its multiplicative semigroup is involutive.*

If $\mathbf{Q}$ is a unital quantale with multiplicative identity element 1 then its subset $\{x \in Q \,|\, x \leq 1\}$ is a subquantale of $\mathbf{Q}$ called the integral part of $\mathbf{Q}$.

Examples

1. Let **R** be an associative (not necessarily commutative) ring. Then the set Id(**R**) of all two-sided ideals of **R** is an integral quantale with respect to the usual join operation (that is, sum) and the multiplication of ideals.

2. Let **L** be a complete relatively pseudocomplemented lattice, in other words, a complete lattice where the meet operation distributes over all joins. Then $(L, \vee, \wedge)$ is a commutative integral quantale. Note that quantales in which the meet operation coincides with multiplication are called **frames**.

Tolerances

Let $\mathbf{A}$ be a (universal) algebra. An n-ary relation R on A is called **compatible** if it is a subuniverse of $\mathbf{A}^n$. Reflexive and symmetric binary relations are called **tolerances**. Tolerance of an algebra $\mathbf{A}$ is a compatible tolerance relation on A.

In what follows we almost exclusively are dealing with complete lattices. Therefore, when speaking about compatible relations on lattices, we always assume the completeness, that is, they must be closed with respect to all joins and meets. In particular, a tolerance of a complete lattice $\mathbf{L}$ is a tolerance relation on L that is closed with respect to all joins and meets.

Quantale of tolerances

E. Bartl, M. Krupka: Residuated lattices of block relations: size reduction of concept lattices. Internat. J. Gen. Syst. **45**, 773–789 (2016)

The set of all tolerances of a complete lattice forms a quantale with the set-theoretical intersection $\cap$ in role of $\vee$ and the multiplication

$$S \otimes T = (S \circ \geq \circ T) \cap (T \circ \leq \circ S) .$$

We will denote this quantale by $\mathbf{Tol}\,(\mathbf{L})$. This quantale is integral with multiplicative identity element Δ. Obviously, the quantale $\mathbf{Tol}\,(\mathbf{L})$ is involutive, too.

Implicitly another quantale structure on the same set appeared in Kaarli-Pixley's book "Polynomial completeness in algebraic systems". Instead of $\otimes$ we had in role of multiplication the operation

$$S * T = \{(x, y) \in L^2 \mid (x \vee y, x \wedge y) \in S \circ T\}.$$

We called the tolerance $S * T$ the *symmetrized product* of S and T. It was proved for the lattices of finite length that $*$ is associative and distributes over intersections. The proofs, however, remain valid for all complete lattices.

Proposition 1. *If S and T are tolerances of a complete lattice then $S \otimes T = T * S$.*

Quantales of reflexive relations

Theorem 1. *Let* $\mathbf{L}$ *be a complete lattice. Then the set of all reflexive binary compatible (complete) relations on* L *forms a quantale with* $\cap$ *in role of* $\vee$ *and* $\circ$ *in role of multiplication.*

Theorem 2. *Let* $\mathbf{A}$ *be a finite majority algebra. Then the set of all reflexive binary compatible relations on* A *forms a quantale with* $\cap$ *in role of* $\vee$ *and* $\circ$ *in role of multiplication.*

We denote the quantales given in these two theorems by $\mathbf{Re}(\mathbf{L})$ and $\mathbf{Re}(\mathbf{A})$, respectively.

There is an example showing that the same statement is not true in case of all (not necessarily complete) lattices and in case of infinite majority algebras.

Valentini's ordered sets

S. Valentini: Representation theorems for quantales. Math. Logic Quart., **40**, 182–190 (1994)

Definition 3. *Following Valentini we say that a binary relation R on* **L** *is* $\leq$-**ordered** *if it satisfies the following conditions:*

1. *given any $u, x, y, z \in L$, if $u \leq x$, $(x, y) \in R$ and $y \leq z$ then $(u, z) \in R$;*

2. *given any $t \in L$ and $A \subseteq L$, if $(a, t) \in R$ for every $a \in A$ then $(\bigvee A, t) \in R$;*

3. *given any $t \in L$ and $A \subseteq L$, if $(t, a) \in R$ for every $a \in A$ then $(t, \bigwedge A) \in R$.*

The $\geq$-**ordered relations** *are defined dually.*

Proposition 2. *Let R be a binary relation on a complete lattice $\mathbf{L}$. Then the following are equivalent:*

1. *R is a $\leq$-ordered relation;*
2. *R is the universe of a complete subdirect square of $\mathbf{L}$ containing the 'corner' element $(0, 1)$;*
3. *R is a compatible binary relation on $\mathbf{L}$ containing the relation $\Gamma(\mathbf{L}) = \{(x, y) \in L^2 \mid x = 0 \ \text{ or } \ y = 1\}$.*

Such subsets of $\mathbf{L}^2$ were studied in KP book and the set of all such subsets was denoted there by $\mathrm{Subd}^{01}(\mathbf{L}^2)$.

Valentini proved that the set of all $\leq$-ordered sets over a complete lattice $\mathbf{L}$ forms a quantale with respect to $\cap$ and $\circ$. We will denote this quantale by $\mathbf{OR}_{\leq}(\mathbf{L})$. The quantale $\mathbf{OR}_{\leq}(\mathbf{L})$ is not integral but only unital. The multiplicative identity element of $\mathbf{OR}_{\leq}(\mathbf{L})$ is the diagonal relation Δ and the integral part of $\mathbf{OR}_{\leq}(\mathbf{L})$ consists of all reflexive $\leq$-ordered relations. We will denote it by $\mathbf{ReOR}_{\leq}(\mathbf{L})$.

Valentini's representation theorem

Let $\mathbf{Q}$ be a quantale. For any $x \in Q$, put $R_x = \{y, z) \in Q^2 \mid xy \leq z\}$.
Theorem 3. (Valentini) *Let* $\mathbf{Q}$ *be a unital quantale with underlying lattice* $\mathbf{L}$*. Then the mapping* $H : \mathbf{Q} \to \mathbf{OR}_{\geq}(\mathbf{L})$*,* $H(x) = R_x$*, is an embedding of quantales.*

Moreover, Valentini described the image of H in $\mathbf{OR}_{\geq}(\mathbf{L})$.

For $x, y \in Q$, we define the "right implication"

$$y \leftarrow x = \bigvee \{w \in Q \mid wx \leq y\} .$$

Theorem 4. (Valentini) *Let* $\mathbf{Q}$ *be a unital quantale with underlying lattice* $\mathbf{L}$*. Then* $H(Q)$ *concists precisely of those* $\geq$*-ordered relations* R *on* L *that satisfy the following condition: 1)* $(y, z) \in R$ *and* $a \in L$ *then* $(ya, za) \in R$ *and* $(y \leftarrow a, z \leftarrow a) \in R$*, too.*

Quantales of join/meet endomorphisms

Let $\mathbf{L}$ be a complete lattice and $\mathrm{End}_\vee(\mathbf{L})$ the set of all join endomorphisms of $\mathbf{L}$. It is easy to check that $\mathrm{End}_\vee(\mathbf{L})$ forms a quantale with pointwise defined join operation in role of $\vee$ and usual composition of maps in role of multiplication. We will denote this quantale by $\mathbf{End}_\vee(\mathbf{L})$. This quantale is obviously unital; the identity map is its multiplicative identity element. But $\mathbf{End}_\vee(\mathbf{L})$ is not integral. Indeed, its greatest element in sense of $\vee$ is the "almost constant" map that fixes 0 and maps the rest to 1. Thus, the integral part of $\mathbf{End}_\vee(\mathbf{L})$ consists of decreasing join endomorphisms of $\mathbf{L}$; we denote it by $\mathbf{End}_{\vee,\downarrow}(\mathbf{L})$.

The quantales of meet endomorohisms $\mathbf{End}_\wedge(\mathbf{L})$ and increasing meet endomorphisms $\mathbf{End}_{\wedge,\uparrow}(\mathbf{L})$ are defined similarly.

Some natural isomorphisms

Theorem 5. (Valentini) *The quantales* $\mathbf{OR}_{\geq}(\mathbf{L})$ *and* $\mathbf{End}_{\vee}(\mathbf{L})$ *are isomorphic.*

Corollary 1. *The quantales* $\mathbf{ReOR}_{\geq}(\mathbf{L})$ *and* $\mathbf{End}_{\vee,\downarrow}(\mathbf{L})$ *are isomorphic.*

These isomorphisms are established by the maps:

$$R \mapsto f^R \;\text{ where }\; f^R(x) = \bigwedge\{z \mid (x,z) \in R\}\,;$$

$$f \mapsto R^f = \{(x,y) \mid f(y) \leq x\}\,.$$

Theorem 6. (Janowitz, 1986) *The quantales* $\mathbf{Tol}(\mathbf{L})$ *and* $\mathbf{End}_{\vee,\downarrow}(\mathbf{L})$ *are isomorphic. This isomorphisms is established by the maps:*

$$T \mapsto f^T \text{ where } f^T(x) = \bigwedge\{z \mid (x,z) \in T\,;$$

$$f \mapsto T^f = \{(x,y) \mid f(x \vee y) \leq x \wedge y\}\,.$$

The similar correspondence between congruences and idempotent decreasing join endomorphisms was established by Janowitz already in 1965. The same correspondence in case of finite lattices appears in the beginning of "Finite algebras" by Hobby and McKenzie.

Order affine complete lattices

In 1977 R. Wille obtained a description of finite order affine complete lattices. In essence, he proved the following theorem.

Theorem 7. *A finite lattice* $\mathbf{L}$ *is order affine complete if and only if the quantale* $\mathbf{End}_{\vee,\downarrow}(\mathbf{L})$ *is generated by its idempotents.*

When writing the KP book on polynomial completeness, we translated this Wille's result into the language of tolerances.

Theorem 8. *A finite lattice* $\mathbf{L}$ *is order affine complete if and only if the quantale* $\mathbf{Tol}(\mathbf{L})$ *is generated by its idempotents, that is, by congruences of* $\mathbf{L}$.

Representation theorems

Given a quantale $\mathbf{Q}$ and $a \in Q$, we introduce the mappings λ_a, $\rho_a : Q \to Q$ defined by $\lambda_a(x) = ax$, $\rho_a(x) = xa$. These mappings are called **left** and **right translations** of $\mathbf{Q}$.

Theorem 9. *Let $\mathbf{Q}$ be an integral quantale with underlying lattice $\mathbf{L}$. Then the mapping $F : \mathbf{Q} \to \mathbf{End}_{\vee,\downarrow}(\mathbf{L})$, $G(a) = \lambda_a$, is an embedding of quantales.*

Theorem 10. *Let $\mathbf{Q}$ be an integral quantale with underlying lattice $\mathbf{L}$. Then:*

- *the mapping $G : \mathbf{Q} \to \mathbf{Tol}(\mathbf{L})$, $G(a) = \{(x, y) \in Q^2 \mid a(x \vee y) \leq x \wedge y\}$, is an embedding of quantales;*
- *$G(Q)$ consists precisely of all such tolerances of $\mathbf{L}$ that $(x, y) \in T$ implies $(xb, yb) \in T$ and $(x \leftarrow b, y \leftarrow b) \in T$, for every $b \in Q$.*

The proof of the Theorem 9 practically repeats the standard proof of Cayley theorem for groups. The Theorems 9 and 10 are actually derivable from the results due to Valentini, using the isomorphisms we described above.

Note that the mapping G is not a lattice embedding. It takes the joins in $\mathbf{Q}$ (same as in $\mathbf{L}$) to intersections in $\mathbf{Tol}(\mathbf{L})$ but does not take meets in $\mathbf{L}$ to joins in $\mathbf{Tol}(\mathbf{L})$. We do have a corresponding counterexample.

Let $\mathbf{A}$ be the algebra we get from $\mathbf{Q}$ by adding to its operations all right translations ρ_a, $a \in Q$. It follows from the previous theorem that $G(Q) \subseteq \mathrm{Tol}(\mathbf{A})$. Actually $\mathbf{Tol}(\mathbf{A})$ is a subquantale of $\mathbf{Tol}(\mathbf{L})$.

Theorem 11. *The mapping G embeds the quantale $\mathbf{Q}$ into $\mathbf{Tol}(\mathbf{A})$. It also maps the meets in $\mathbf{L}$ to joins in $\mathbf{Tol}(\mathbf{A})$.*

Corollary 2. *Let* $\mathbf{L}$ *be a complete relatively pseudocomplemented lattice. Then* G *embeds the lattice* $\mathbf{L}$ *into the lattice of (complete) congruences of* $\mathbf{L}$*.*

Theorem 12. *For every finite integral involutive quantale* $\mathbf{Q}$ *there exist a finite majority algebra* $\mathbf{A}$ *such that* $\mathbf{Q}$ *is isomorphic to* $\mathbf{Re}(\mathbf{A})$*.*

Underlying lattice of a finite quantale

Definition 4. *Let* $\mathbf{L}$ *be a lattice with* 0*. A pair of elements* $a, b \in L$ *is called:*

- **disjoint** *if* $a \wedge b = 0$*;*
- **distributive** *if* $c \wedge (a \vee b) = (c \wedge a) \vee (c \wedge b)$ *holds for any* $c \in L$*.*

The lattice $\mathbf{L}$ *is called*

- **pseudocomplemented** *if for each* $x \in L$ *there exists an element* $x^* \in L$ *(the* **pseudocomplement** *of* x*) such that for any* $y \in L$*,* $y \wedge x = 0 \Leftrightarrow y \leq x^*$*.*
- **distributive in** 0 *if all disjoint pairs of* $\mathbf{L}$ *are distributive.*

Corresponding dual notions are defined for lattices with 1*.*

Theorem 13. *The underlying lattice of any finite integral quantale is dually pseudocomplemented and distributive in* 1.

In the proof of this theorem some earlier results by Czédli, Horváth and Radeleczki were applied.

THANK YOU!