The Extensions of Green's Relations on Upper Triangular Tropical Matrices

Munazza Naz

University of York

NSAC, 14-18 June 2017

Joint work with Victoria Gould

Contents

- Tropical Semifield
- Tropical Algebra
- Upper Triangular Tropical Matrices $U_n(\mathbb{FT})$
- Green's Equivalence Relations on $U_n(\mathbb{FT})$
- Idempotent and Regular Matrices in $U_n(\mathbb{FT})$
- $*-$ and $\sim -$ Extensions of Green's Relations on $U_n(\mathbb{FT})$

Tropical Semifield

The finitary tropical (or max-plus) semifield $\mathbb{FT}$ has elements from $\mathbb{R}$ with binary operations defined as:

$$\begin{aligned} x \oplus y &= max(x, y); \text{ and} \\ x \otimes y &= x + y. \end{aligned}$$

We see that $(\mathbb{FT}, \oplus, \otimes)$ is an idempotent semifield.

Its generalisations are $(\mathbb{T}, \oplus, \otimes)$ and $(\overline{\mathbb{T}}, \oplus, \otimes)$, where $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ and $\overline{\mathbb{T}} = \mathbb{R} \cup \{-\infty, \infty\}$.

Conventions

Here we have some conventions:

$$\begin{aligned}
a \oplus -\infty &= a = -\infty \oplus a, \\
a \oplus \infty &= \infty = \infty \oplus a, \\
\infty \oplus -\infty &= \infty = -\infty \oplus \infty, \\
a \otimes -\infty &= -\infty = -\infty \otimes a, \\
a \otimes \infty &= \infty = \infty \otimes a, \\
\infty \otimes -\infty &= -\infty = -\infty \otimes \infty,
\end{aligned}$$

for all $a \in \mathbb{FT}$.

Tropical Matrices and Vectors

Let $M_n(\mathbb{S})$ denotes the set of all $n \times n$ matrices over $\mathbb{S} \in \{\mathbb{FT}, \mathbb{T}, \overline{\mathbb{T}}, \}$, with multiplication $\otimes$ defined as:

$$(A \otimes B)_{ij} = \bigoplus_{k=1}^{n} (A_{ik} \otimes B_{kj})$$

for all $A, B \in M_n(\mathbb{S})$.
Then $(M_n(\mathbb{S}), \otimes)$ is a semigroup.

Tropical Affine n-Space

Let $\mathbb{S}^n$ denote the set of all real n-tuples $\vec{v} = (v_1, ..., v_n)$ with obvious operations of addition and scalar multiplication:

$$(\vec{v} \oplus \vec{w})_i = v_i \oplus w_i,$$

$$(\lambda \otimes \vec{v})_i = \lambda \otimes v_i.$$

Semigroup of Tropical Matrices

For $A \in M_n(\mathbb{S})$

- The *row space* $R(A) \subseteq \mathbb{S}^n$ is defined as the tropical submodule of $\mathbb{S}^n$ generated by the rows of A.

And similarly,

- The column space $C(A) \subseteq \mathbb{S}^n$ is defined as the tropical submodule $\mathbb{S}^n$ generated by the columns of A.

Example

Let

$$A = \begin{bmatrix} 2 & 4 & 5 \\ 5 & 3 & 7 \\ 9 & \infty & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 7 & 6 & 7 \\ -\infty & 4 & 9 \\ 6 & 3 & 3 \end{bmatrix}.$$

Then,

$$AB = \begin{bmatrix} 11 & 8 & 13 \\ 13 & 11 & 12 \\ 16 & \infty & \infty \end{bmatrix}.$$

Questions about Semigroup of Tropical Matrices

Natural questions about the multiplicative structure of semigroup of tropical matrices arise:

- Ideals
- Green's relations

A series of articles describing these concepts and using geometric arguments is available:

1. M. Johnson and M. Kambites, Multiplicative structure of 2×2 tropical matrices, Linear Algebra and its Applications, 435(2011), pp. 1612-1625.
2. C. Hollings and M. Kambites, Tropical matrix duality and Green's $\mathcal{D}$ relation, Journal of the London Mathematical Society, 86(2012), pp. 520-538.
3. M. Johnson and M. Kambites, Green's $\mathcal{J}$ -order and the rank of tropical matrices, Journal of Pure and Applied Algebra, 217(2013), pp. 280-292.

Questions about Semigroup of Tropical Matrices

- Idempotents
- Subgroups
- Regularity

1. M. Kambites and M. Johnson, Idempotent tropical matrices and finite metric spaces, Advances in Geometry, 14(2014), pp. 253-276.
2. M. Johnson and M. Kambites, Convexity of tropical polytopes, Linear Algebra and its Applications, 485(2015), pp. 531-544.
3. Z. Izhakian, M. Johnson, and M. Kambites, Pure dimension and projectivity of tropical polytopes, Advances in Mathematics, 303(2016), pp. 1236-1263.

Theorem

Let $A, B \in M_n(\mathbb{S})$ *for* $\mathbb{S} \in \{\mathbb{FT}, \mathbb{T}, \mathbb{T}\}$. *Then*

1. $A \preceq_{\mathcal{L}} B \Leftrightarrow R_{\mathbb{S}}(A) \subseteq R_{\mathbb{S}}(B)$;
2. $A\mathcal{L}B \Leftrightarrow R_{S}(A) = R_{\mathbb{S}}(B)$;
3. $A \preceq_{\mathcal{R}} B \Leftrightarrow C_{\mathbb{S}}(A) \subseteq C_{\mathbb{S}}(B)$;
4. $A\mathcal{R}B \Leftrightarrow C_{\mathbb{S}}(A) = C_{\mathbb{S}}(B)$;
5. $A\mathcal{H}B \Leftrightarrow C_{\mathbb{S}}(A) = C_{\mathbb{S}}(B)$ *and* $R_{\mathbb{S}}(A) = R_{\mathbb{S}}(B)$;
6. $A\mathcal{D}B$ *if and only if* $C_{\mathbb{S}}(A)$ *and* $C_{\mathbb{S}}(B)$ *are isomorphic as* $\mathbb{S}$*-modules;*
7. $A\mathcal{D}B$ *if and only if* $R_{\mathbb{S}}(A)$ *and* $R_{\mathbb{S}}(B)$ *are isomorphic as* $\mathbb{S}$*-modules.*

C. Hollings and M. Kambites, Tropical matrix duality and Green's D relation, Journal of the London Mathematical Society, 86 (2012), pp. 520-538.

Upper Triangular Tropical Matrices

Let $U_n(\mathbb{T})$ be the subset of all *upper triangular tropical matrices* in $M_n(\mathbb{T})$, where $M_{ij} = -\infty$ for all $i > j$ and $M \in U_n(\mathbb{T})$.

- This set $U_n(\mathbb{T})$ is a subsemigroup of $M_n(\mathbb{T})$ under its operation of multiplication;
- The set of matrices in $U_n(\mathbb{T})$ where all entries on and above the diagonal are finite forms a subsemigroup of $U_n(\mathbb{T})$, which we denote by $U_n(\mathbb{FT})$;
- Many structural properties of upper triangular tropical matrix semigroups have been worked out with details in Taylor's thesis and some articles are also available on semigroup identities which hold in triangular case

References

1. Z. Izhakian, Semigroup identities in the monoid of triangular tropical matrices, Semigroup Forum, 88 (2014), no. 1, 145-161.
2. J. Okninski, Identities of the semigroup of upper triangular tropical matrices. Communications in Algebra, 43 (2015), pp. 4422-4426.
3. M. Taylor, On upper triangular tropical matrix semigroups, tropical matrix identities and T-modules, Thesis submitted to the University of Manchester (2016).
4. Z. Izhakian, Erratum to: Semigroup identities in the monoid of triangular tropical matrices, Semigroup Forum, 92 (2016), p733.
5. L. Daviaud, M. Johnson, M. Kambites, Identities in Upper Triangular Tropical Matrix Semigroups and the Bicyclic Monoid, preprint, (2016).

Known Results

In his thesis, Taylor has shown following results:

- Every $M \in U_n(\mathbb{FT})$ has both row and column rank n
- For $M, N \in U_n(\mathbb{FT})$, $M\mathcal{R}N$ (respectively $M\mathcal{L}N$) if and only if the ith row (column) of N is a scaling of the ith row (column) of M.
- Green's relations for $U_n(\mathbb{FT})$ are the restrictions of corresponding relations on $M_n(\mathbb{T})$
- $U_2(\mathbb{FT})$ has only one $\mathcal{D}$-class, in fact, $U_2(\mathbb{FT})$ is an inverse semigroup
- $U_n(\mathbb{FT})$ has only one $\mathcal{J}$-class, for all n.

Comments and Examples

$U_2(\mathbb{T})$ is not an inverse semigroup.
Recall that a semigroup S is inverse if and only if it is regular with commuting idempotents.

Example

Let

$$E = \begin{bmatrix} 0 & -\infty \\ 0 & -\infty \end{bmatrix}, F = \begin{bmatrix} -\infty & -\infty \\ 0 & 0 \end{bmatrix}.$$

Then E, F are idempotents in $U_n(\mathbb{T})$ and

$$EF = \begin{bmatrix} -\infty & -\infty \\ -\infty & -\infty \end{bmatrix} \neq \begin{bmatrix} -\infty & -\infty \\ 0 & -\infty \end{bmatrix} = FE.$$

Comments and Examples

1. $U_n(\mathbb{FT})$ is not regular for $3 \leq n$.

Example

Let $M \in U_3(\mathbb{FT})$

$$M = \begin{bmatrix} 0 & 1 & 0 \\ -\infty & 0 & 2 \\ -\infty & -\infty & 0 \end{bmatrix}$$

If there is some $N \in U_3(\mathbb{FT})$ such that $MNM = M$, then we must have $N_{11} = N_{22} = N_{33} = 0$ and

$$\begin{aligned} \max\{1, N_{12}\} &= 1; \\ \max\{2, N_{23}\} &= 2\ ; \\ \max\{N_{13}, (1 + N_{23}), 3, (2 + N_{12})\} &= 0, \end{aligned}$$

which is not possible for any choice of N. Thus M is not regular.

Extension of Green's Relations

On a semigroup S, the relation $\leq_{\mathcal{R}^*}$ is defined by the rule that for $a, b \in S$, $a \leq_{\mathcal{R}^*} b$ if and only if

$$xb = yb \Rightarrow xa = ya \ \text{ and } \ xb = b \Rightarrow xa = a$$

for all $x, y \in S$. Clearly $\leq_{\mathcal{R}^*}$ is a quasi-order on S. The equivalence relation associated with $\leq_{\mathcal{R}^*}$ is denoted by $\mathcal{R}^*$ and it is a left congruence on S. The relation $\mathcal{L}^*$ is defined dually and the relations $\mathcal{H}^*$ and $\mathcal{D}^*$ by putting

$$\mathcal{H}^* = \mathcal{R}^* \cap \mathcal{L}^*; \ \mathcal{D}^* = \mathcal{R}^* \vee \mathcal{L}^*.$$

Extension of Green's Relations

It follows, $a \mathcal{R} b$ implies that $a \mathcal{R}^* b$ and, $e \mathcal{R} f$ if and only if $e \mathcal{R}^* f$ for all $e, f \in E(S)$. For any $a \in S$, the set of left identities for a in $E \subseteq E(S)$ is denoted by ${}_E a$, that is,

$$ {}_E a = \{ e \in E : ea = a \}. $$

The relation $\leq_{\widetilde{\mathcal{R}}}$ on S is defined by the rule that for $a, b \in S$, $a \leq_{\widetilde{\mathcal{R}}} b$ if and only if for all $e \in E$

$$ eb = b \Rightarrow ea = a. $$

The equivalence relation associated with $\leq_{\widetilde{\mathcal{R}}}$ is denoted by $\widetilde{\mathcal{R}}_E$. We have following result:

Lemma

For any $a \in S$ and $e \in E$, $a \, \widetilde{\mathcal{R}} \, e$ if and only if $ea = a$ and for all $f \in E$, if $fa = a$ then $fe = e$.

Extension of Green's Relations

We say that S is

- left E-abundant if every $\mathcal{R}^*$-class contains an idempotent of E;
- Weakly left E-abundant if every $\widetilde{\mathcal{R}}$-class contains an idempotent of E;
- *abundant* if each $\mathcal{R}^*$-class and each $\mathcal{L}^*$-class of S contains an idempotent;
- A semigroup S is *weakly abundant* (Fountain Semigroup!) if each $\widetilde{\mathcal{R}}$-class and each $\widetilde{\mathcal{L}}$-class of S contains an idempotent.

For basic facts about the relations about abundant and weakly abundant semigroups we refer to:

1. J. Fountain, A class of right PP monoids, Quart. J. Math. Oxford (2) 28 (1977), pp 285-300.
2. J. Fountain, Adequate semigroups, Proc. Edinb. Math. Soc. (2) 22 (1979), pp 113-125.

Idempotents in $U_n(\mathbb{FT})$

A matrix E is an idempotent in $U_n(\mathbb{FT})$ exactly if

$$E_{ij} = \bigoplus_{k=1}^{n} (E_{ik} \otimes E_{kj})$$

But this says that:

$$\begin{aligned} E_{ii} + E_{ii} &= E_{ii} \text{ or } E_{ii} = 0 \text{ for each } 1 \leq i \leq n \quad \text{and} \\ E_{ik} \otimes E_{kj} &\leq E_{ij} \text{ for each } k,\ 1 \leq i < k < j \leq n. \end{aligned}$$

Idempotents in $U_n(\mathbb{FT})$

Theorem

For $E, F \in \mathbf{E}(U_n(\mathbb{FT}))$,

$$(EF)^{[\frac{n-1}{2}]}E = (FE)^{[\frac{n-1}{2}]}F.$$

Corollary

For $E, F \in \mathbf{E}(U_n(\mathbb{FT}))$,

$$(EF)^{[\frac{n+1}{2}]} = (FE)^{[\frac{n+1}{2}]}.$$

Corollary

For $E, F \in \mathbf{E}(U_3(\mathbb{FT}))$,

$$(EF)^2 = (FE)^2.$$

Idempotents in $U_n(\mathbb{FT})$

Theorem

Let

$$\overline{\mathbf{E}}(U_n(\mathbb{FT})) = \{X \in U_n(\mathbb{FT}) | X_{ii} = 0, 1 \leq i \leq n\}.$$

Then $\overline{\mathbf{E}}(U_n(\mathbb{FT}))$ *is a subsemigroup and*

$$\overline{\mathbf{E}}(U_n(\mathbb{FT})) = (\mathbf{E}(U_n(\mathbb{FT})))^{(n-1)}.$$

Corollary

For $X \in \overline{\mathbf{E}}(U_n(\mathbb{FT}))$, $\{X, X^2, ..., X^{n-1}\} \cap \mathbf{E}(U_n(\mathbb{FT}))$ *is non-empty.*

Left Identities

Lemma

Let $A \in U_n(\mathbb{FT})$. Then for any $X \in U_n(\mathbb{FT})$, $XA = A$ exactly if

$$X_{ij} \leq \min_{j \leq k \leq n} (A_{ik} - A_{jk}),\ 1 \leq i < j \leq n \text{ and } X_{ii} = 0,\ 1 \leq i \leq n.$$

Thus, every $A \in U_n(\mathbb{FT})$ can be associated with a set of matrices denote $_{\overline{\mathbf{E}}}\mathbf{A}$ acting as left identities of A and matrix $_AE \in U_n(\mathbb{FT})$ whose diagonal elements are zero and

$$({}_AE)_{ij} = \min_{j \leq k \leq n} (A_{ik} - A_{jk}),\ 1 \leq i < j \leq n.$$

Weakly Abundant

Lemma

For every $A \in U_n(\mathbb{FT})$ there exists a unique idempotent matrix in $U_n(\mathbb{FT})$ denoted ${}_AE$ such that ${}_AEA = A$ and if $FA = A$ for some $F \in U_n(\mathbb{FT})$ then $F_AE =_A E$, in other words, $A \, \widetilde{\mathcal{R}} \, {}_AE$.

Theorem

$U_n(\mathbb{FT})$ is right(left) weakly abundant and therefore a Fountain Semigroup which is not regular (Not Abundant too).

Example for Non-compatibility

Example

Let

$$A = \begin{bmatrix} 2 & 4 & 5 \\ -\infty & 3 & 7 \\ -\infty & -\infty & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 7 & 6 & 7 \\ -\infty & 4 & 9 \\ -\infty & -\infty & 3 \end{bmatrix}.$$

But, no value of t in $\mathbb{FT}$ can give $4 \otimes t = 6$; $3 \otimes t = 4$.Thus $C(A) isnotequalto C(B)$.

*- Relation

Theorem

Let $A, B \in U_n(\mathbb{FT})$. Then $CA \, \widetilde{\mathcal{R}} \, CB$ for all $C \in U_n(\mathbb{FT})$ if and only if $A \, \mathcal{R} \, B$.

Theorem

Let $A, B \in U_n(\mathbb{FT})$. Then $A \, \mathcal{R} \, B$ exactly if $A \, \mathcal{R}^* \, B$.

Thank You!